

# Algebraic properties of Manin matrices II: $q$ -analogues and integrable systems.

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## Abstract

We study a natural  $q$ -analogue of a class of matrices with noncommutative entries, which were first considered by Yu. I. Manin in 1988 in relation with quantum group theory, (called *Manin Matrices* in [5]) . These matrices we shall call  $q$ -Manin matrices(qMMs). They are defined, in the  $2 \times 2$  case, by the relations

$$M_{21}M_{12} = qM_{12}M_{21}, \quad M_{22}M_{12} = qM_{12}M_{22}, \quad [M_{11}, M_{22}] = q^{-1}M_{21}M_{12} - qM_{12}M_{21}.$$

They were already considered in the literature, especially in connection with the  $q$ -Mac Mahon master theorem [16], and the  $q$ -Sylvester identities [25]. The main aim of the present paper is to give a full list and detailed proofs of algebraic properties of qMMs known up to the moment and, in particular, to show that most of the basic theorems of linear algebras (e.g., Jacobi ratio theorems, Schur complement, the Cayley-Hamilton theorem and so on and so forth) have a straightforward counterpart for  $q$ -Manin matrices. We also show how this class of matrices fits within the theory of quasi-determinants of Gel'fand-Retakh and collaborators (see, e.g., [17]). In the last sections of the paper, we frame our definitions within the tensorial approach to non-commutative matrices of the Leningrad school, and we show how the notion of  $q$ -Manin matrix is related to theory of Quantum Integrable Systems.

**Key words:** Noncommutative determinant, quasideterminant, Manin matrix, Jacobi ratio theorem, Newton identities, Cayley-Hamilton theorem, Schur complement, Dodgson's condensation, Lax matrix, R-matrix.

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b><math>q</math>-Manin matrices</b>	<b>5</b>
2.1	Definition of $q$ -Manin matrices . . . . .	5
2.2	Linear transformations of $q$ -(anti)-commuting variables . . . . .	7
2.3	Elementary properties . . . . .	9
2.4	Relations with quantum groups . . . . .	10
2.5	Hopf structure . . . . .	11
<b>3</b>	<b>The <math>q</math>-determinant and Cramer's formula</b>	<b>12</b>
3.1	The $q$ -determinant . . . . .	12
3.1.1	Definition of the $q$ -determinant . . . . .	12
3.1.2	The $q$ -Grassmann algebra . . . . .	13
3.1.3	The $q$ -determinant, $q$ -minors and the $q$ -Grassmann algebra . . . . .	15
3.2	Properties of the $q$ -determinants of $q$ -Manin matrices . . . . .	16
3.3	The $q$ -Characteristic polynomial . . . . .	21
3.4	A $q$ -generalization of the Cramer formula and quasi-determinants . . . . .	22
3.4.1	Left adjoint matrix . . . . .	22
3.4.2	Relation with the quasi-determinants . . . . .	23
3.4.3	Gauss decomposition and $q$ -determinant . . . . .	25
<b>4</b>	<b><math>q</math>-Minors of a <math>q</math>-Manin matrix and its inverse</b>	<b>26</b>
4.1	Jacobi ratio theorem . . . . .	27
4.1.1	Preliminary propositions . . . . .	27
4.1.2	Jacobi ratio theorem for $q$ -Manin matrices . . . . .	29
4.2	Corollaries . . . . .	31
4.2.1	Lagrange-Desnanot-Jacobi-Lewis Carroll formula . . . . .	31
4.2.2	The inverse of a $q$ -Manin matrix is a $q^{-1}$ -Manin matrix . . . . .	32
4.2.3	Schur complements . . . . .	33
4.2.4	Sylvester's theorem . . . . .	34
4.3	Plücker relations: an example . . . . .	35
<b>5</b>	<b>Tensor approach to <math>q</math>-Manin matrices</b>	<b>36</b>
5.1	Leningrad tensor notations . . . . .	36
5.2	Tensor relations for $q$ -Manin matrices . . . . .	37
5.2.1	Pyatov's Lemma . . . . .	37
5.2.2	Higher $q$ -(anti)-symmetrizers and $q$ -Manin matrices . . . . .	39
5.3	The $q$ -determinant and the $q$ -minors as tensor components . . . . .	40
5.3.1	Action of the $q$ -anti-symmetrizer in the basis $\{e_{j_1, \dots, j_m}\}$ . . . . .	41
5.3.2	Components of the tensor $A_m^q M^{(1)} \cdots M^{(m)}$ . . . . .	41
5.4	$q$ -powers of $q$ -Manin matrices. Cayley-Hamilton theorem and Newton identities . . . . .	43
5.4.1	Cayley-Hamilton theorem . . . . .	43

5.4.2	Newton theorem . . . . .	45
5.5	Inverse to a $q$ -Manin matrix . . . . .	46
<b>6</b>	<b>Integrable systems. <math>L</math>-operators</b>	<b>48</b>
6.1	$L$ -operators and $q$ -Manin matrices . . . . .	48
6.2	Quantum determinant of $L$ -operator . . . . .	49
6.3	Generating functions for the commuting operators . . . . .	51
6.4	Quantum powers for $L$ -operators . . . . .	54
<b>A</b>	<b>Proof of Lemma 6.4</b>	<b>55</b>
<b>B</b>	<b>An alternative proof of the Lagrange-Desnanot-Jacobi-Lewis Caroll formula</b>	<b>56</b>

# 1 Introduction

It is well-known that matrices with generically noncommutative elements play a basic role in the theory of quantum integrability as well as other fields of Mathematical Physics. In particular the best known occurrences of such matrices (see, e.g, [8, 14] and the references quoted therein) are the theory of quantum groups, the theory of critical and off-critical phenomena in statistical mechanics models, and various instances of combinatorial problems. More recently, applications of such structures in the realm of the theory of Painlevé equations was considered [36]. The present paper builds, at least in its bulk, on the results of [5]. In that paper, a specific class of non-commutative matrices (that is, matrices with entries in a non commutative algebra – or ring) were considered, namely the so-called Manin matrices. This class of matrices, which are nothing but matrices representing linear transformations of polynomial algebra generators, were first introduced in the seminal paper [30] by Yu. I. Manin, but they attracted sizable attention only recently (especially in the problem of the so-called quantum spectral curve in the theory of Gaudin models [10]). The defining relations for a (column) Manin matrix  $M_{ij}$  are:

1. Elements in the same column commute;
2. Commutators of the cross terms are equal:  $[M_{ij}, M_{kl}] = [M_{kj}, M_{il}]$  (e.g.  $[M_{11}, M_{22}] = [M_{21}, M_{12}]$ ).

The basic claim, fully proven in [5] is that most theorems of linear algebra hold true for Manin matrices (in a form *identical* to that of the commutative case) although the set of Manin matrices is fairly different from that of ordinary matrices – for instance they do not form a ring over the base field. Moreover in some examples the converse is also true, that is, Manin matrices are the most general class of matrices such that “ordinary” linear algebra holds true for them. Apart from that, it is important to remark that the Manin matrix structure appears, as it was pointed out in [4] in many instances related with the theory of Quantum integrable systems, namely the theory of integrable spin chains associated with *rational*  $R$  matrices. The basic aim of the present work is to extend our

analysis to a class of matrices, which we call *q-Manin matrices*, whose defining relations are in some sort the *q*-analogues of the ones for Manin matrices. Namely, they read as:

1. Entries of the same column *q*-commute, i.e.,  $M_{ij}M_{kj} = q^{-1}M_{kj}M_{ij}$
2. The *q*-cross commutation relations:  $M_{ij}M_{kl} - M_{kl}M_{ij} = q^{-1}M_{kj}M_{il} - qM_{il}M_{kj}$ , ( $i < k, j < l$ ) hold.

These defining relations, which, as an expert reader has surely already remarked, can be considered as a "half" of the defining relations for the *Quantum Matrices*, that is, for the elements of the quantum matrix group  $GL_q(N)$ . These matrices were already introduced in the literature; indeed they were considered in the papers [16, 27] (under the name of *right quantum matrices*), where the *q*-generalization of the Mac-Mahon master theorem was proven in connection with the boson-fermion correspondence of quantum physics. Their results (as well as the analysis of the *q*-Cartier Foata type of matrices) were generalized in [25]. A multiparameter right-quantum analogue of Sylvester's identity was found in [24], also in connection with [22]. It is worthwhile to remark that a super-version of the Mac-Mahon master theorem as well as other interesting consequences thereof was discussed in [34].

We would like also to remark that in works of Gel'fand, Retakh and their co-authors (see, e.g., [17, 18, 20]) a comprehensive linear algebra theory for *generic* matrices with noncommutative entries was extensively developed. Their approach is based on the fundamental notion of *quasi-determinant*. Indeed, in the "general non-commutative case" there is no natural definition of a "determinant" (one has  $n^2$  "quasi-determinants" instead) and, quite often, analogues of the proposition of "ordinary" linear algebra are sometimes formulated in a completely different way. Nevertheless it is clear that their results can be fruitfully specialized and applied to some questions here; indeed, in Section 3 we make contact with this theory, and, in Appendix B we give an example of how this formalism can be applied to *q*-Manin matrices. However, in the rest of the paper, we shall not use the formalism of Gel'fand and Retakh, preferring to stress the similarities between our case of *q*-Manin matrices with the case of ordinary linear algebra.

Here we shall give a systematic analysis of *q*-Manin matrices, starting from their basic definitions, with an eye towards possible application to the theory of *q*-deformed quantum integrable systems (that is, in the Mathematical Physics parlance, those associated with *trigonometric R*-matrices). The detailed layout of the present paper is the following:

We shall start with the description of some elementary properties directly stemming from the definition of *q*-Manin matrix in Section 2. In particular, we shall point out that *q*-Manin matrices can be considered both as defining a (left) action on the generators of a *q*-algebra, and as defining a (right) action on the generators of a *q*-Grassmann algebra. We would like to stress that these properties, albeit elementary, will be instrumental in the direct extension of the "ordinary" proofs of some determinantal properties to our case. In Section 3 and 4, after having considered the relations/differences of the *q*-Manin matrices with Quantum group theory, we shall recall in Section the definition of the *q*-determinant and of *q*-minor of a *q*-Manin matrix. In particular, we shall state and prove the analogue of the Cauchy-Binet theorem about the multiplicative property of the *q*-determinant. We

shall limit ourselves to consider the simplest case (of two  $q$ -Manin matrices with the elements of the first commuting with those of the second). We would like to point out that a more ample discussion of (suitable variants of) the Cauchy Binet formula and the Capelli identity in the noncommutative case can be found, besides [5], in [9] and in the recent preprint [3].

Moreover, we shall discuss how the Cramer's rule can be generalized in our case, and prove the Jacobi ratio theorem and the Lagrange-Desnanot-Jacobi-Lewis Carroll formula (also known as Dodgson *condensation* formula). Also, we shall anticipate how the notion of  $q$ -characteristic polynomial (whose properties will be further discussed in Section 5) for a  $q$ -Manin matrix  $M$  should be defined. It is worthwhile to note that this is a specific case in which the  $q$ -generalization of some "ordinary" object requires a suitable choice among the classically equivalent possible definitions. Indeed, the meaningful notion of  $q$ -characteristic polynomial is referred to the weighted sum of the principal  $q$ -minors of  $M$ .

Then we shall show a fundamental property of  $q$ -Manin matrices, that is, closedness under matrix ( $q$ )-inversion. We shall then state the analogue of Schur's complement theorem as well as the Sylvester identities. Further, a glance to the  $q$ -Plücker relations will be addressed (see, for a full treatment in the case of generic non-commutative matrices, the paper [26]).

The aim of Section 5 is twofold. First we shall show, making use of the so-called *Pyatov's Lemma*, how the theory of  $q$ -Manin matrices of rank  $n$  can be framed within the tensor approach of the Leningrad School, i.e. interpreting  $q$ -minors and the  $q$ -determinant as suitable elements in the tensor algebra of  $\mathbb{C}^n$ . This approach will be helpful in reaching the second aim of the section, that is, establishing the Cayley-Hamilton theorem and the Newton identities for  $q$ -Manin matrices.

Section 6 shows, using the formalism introduced in Section 5, how the theory of  $q$ -Manin matrices fits within the scheme of the Quantum Integrable spin systems of trigonometric type. We shall first show that, if  $L(z)$  is a Lax matrix satisfying the Yang-Baxter  $RLL = LLR$  relations (Eq. 6.5) with a trigonometric  $R$ -matrix  $R$ , then the matrix

$$M = L(z)q^{-2z\frac{\partial}{\partial z}},$$

is actually a  $q$ -Manin matrix, and, moreover, that the *quantum determinant* of  $L(z)$  defined by the Leningrad School (see, e.g., [13]) coincides with the  $q$ -determinant of the associated  $q$ -Manin matrix  $M$ . Then, generalizing the corresponding construction of [4, 5], we shall show how an alternative set of quantum mutually commuting quantities can be obtained considering (ordinary) traces of suitably defined "quantum" powers of  $M$ .

The Appendix contains two subsections. The first is devoted to the detailed proof of the most technical Lemma of Section 5, which we extensively use in Section 6. In the second, we shall give an alternative proof of the Dodgson condensation formula making use of the formalism similar to that of quasi-determinants.

As a general strategy to keep the present paper within a reasonable size, we shall present detailed proofs only in the case when these proofs are substantially different from the "non- $q$ " (that is,  $q \neq 1$ ) case. Otherwise, we shall refer to the corresponding proofs in [5].

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## 2 $q$ -Manin matrices

### 2.1 Definition of $q$ -Manin matrices

Let  $M$  be a  $n \times m$  matrix with entries  $M_{ij}$  in an associative algebra  $\mathfrak{R}$  over  $\mathbb{C}$  and let  $q$  be a non-zero complex number. The algebra is not commutative in general, so that the matrix  $M$  has non-commutative entries.

**Definition 1.** *The matrix  $M$  is called a  $q$ -Manin matrix if the following conditions hold true:*

1. *Entries of the same column  $q$ -commute with each other according to the order of the row indices:*

$$\forall j, i < k : \quad M_{ij}M_{kj} = q^{-1}M_{kj}M_{ij}, \quad (2.1)$$

2. *The cross commutation relations:*

$$\forall i < k, j < l : \quad M_{ij}M_{kl} - M_{kl}M_{ij} = q^{-1}M_{kj}M_{il} - qM_{il}M_{kj} \quad (2.2)$$

*hold.*

The defining relations for  $q$ -Manin matrices can be compactly written as follows:

$$M_{ij}M_{kl} - q^{\text{sgn}(i-k)}q^{-\text{sgn}(j-l)}M_{kl}M_{ij} = q^{\text{sgn}(i-k)}M_{kj}M_{il} - q^{-\text{sgn}(j-l)}M_{il}M_{kj}, \quad (2.3)$$

for all  $i, k = 1, \dots, n$ ,  $j, l = 1, \dots, m$ , where we use the notation

$$\text{sgn}(k) = \begin{cases} +1, & \text{if } k > 0; \\ 0, & \text{if } k = 0; \\ -1, & \text{if } k < 0. \end{cases} \quad (2.4)$$

Indeed, for  $j = l$  and  $i \neq k$  one gets the column  $q$ -commutativity (2.1), while for  $j \neq l$  and  $i \neq k$  one gets the cross commutation relations (2.2) and for  $i = k$  one gets an identity.

**Remark 2.1.** Definition 1 can be reformulated as follows: the matrix  $M$  is  $q$ -Manin matrix if and only if each  $2 \times 2$  submatrix is a  $q$ -Manin matrix. More explicitly, for

any  $2 \times 2$  submatrix  $(M_{(ij)(kl)})$ , consisting of rows  $i$  and  $k$ , and columns  $j$  and  $l$  (where  $1 \leq i < k \leq n$ , and  $1 \leq j < l \leq m$ )

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & M_{ij} & \dots & M_{il} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & M_{kj} & \dots & M_{kl} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \equiv \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & a & \dots & b & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & c & \dots & d & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad (2.5)$$

the following commutation relations hold:

$$ca = qac \quad (q\text{-commutation of the entries in the } j\text{-th column}), \quad (2.6)$$

$$db = qbd \quad (q\text{-commutation of the entries in the } l\text{-th column}), \quad (2.7)$$

$$ad - da = q^{-1}cb - qbc \quad (\text{cross commutation relation}). \quad (2.8)$$

### Examples:

- Let  $\mathfrak{R}$  be the algebra generated by  $a, b, c, d$  over  $\mathbb{C}$  with the relations (2.6), (2.7), (2.8) (or be an algebra containing some elements  $a, b, c, d$  satisfying these relations). Then the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.9)$$

is a  $q$ -Manin matrix (over  $\mathfrak{R}$ ).

- Let us consider  $n$  elements  $x_i \in \mathfrak{R}$ ,  $i = 1, \dots, n$  that  $q$ -commute:  $x_i x_j = q^{-1} x_j x_i$  for  $i < j$  (e.g. we can consider the algebra  $\mathfrak{R}$  generated by  $x_i$  with these relations). Then the column-matrix

$$M = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad (2.10)$$

can be considered a  $q$ -Manin matrix.

- The elements from the same row of a  $q$ -Manin matrix are not required to satisfy any relations. So an arbitrary  $1 \times n$  matrix  $M = (r_1, \dots, r_n)$  can be considered as a  $q$ -Manin matrix.
- If some elements  $m_i$   $q$ -commute, i.e.  $m_i m_j = q^{-1} m_j m_i$ ,  $i < j$ , then the matrix

$$\begin{pmatrix} m_1 & m_1 \\ m_2 & m_2 \\ \dots & \dots \\ m_n & m_n \end{pmatrix}, \quad (2.11)$$

is a  $q$ -Manin matrix. The cross-commutation relation (2.2) follows from  $q$ -commutativity in this case. Moreover, if  $q \neq -1$  then the cross-commutation relations (2.2) for this matrix implies  $q$ -commutativity of  $m_i$ .



- Consider elements  $x_i, r_i \in \mathfrak{R}$ ,  $i = 1, \dots, n$ , such that  $x_i x_j = q^{-1} x_j x_i$  for  $i < j$  and  $[x_i, r_j] = 0 \forall i, j$ . Then the matrix

$$M = \begin{pmatrix} x_1 r_1 & x_1 r_2 & \dots & x_1 r_m \\ x_2 r_1 & x_2 r_2 & \dots & x_2 r_m \\ \dots & \dots & \dots & \dots \\ x_n r_1 & x_n r_2 & \dots & x_n r_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \begin{pmatrix} r_1 & r_2 & \dots & r_m \end{pmatrix}. \quad (2.12)$$

is a  $q$ -Manin matrix. This fact can be easily checked by direct calculation.

- We refer to [4], [5], [7], [39] for examples of  $q$ -Manin matrices for  $q = 1$  related to integrable systems, Lie algebras, etc. In the  $q = 1$  case the  $q$ -Manin matrices are simply called *Manin matrices*.
- Let  $\mathbb{C}[x, y]$  be the algebra generated by  $x, y$  with the relation  $yx = qxy$ . Consider the operators  $\partial_x, \partial_y: \mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]$  of the corresponding differentiations:  $\partial_x(x^n y^m) = nx^{n-1}y^m$ ,  $\partial_y(y^m x^n) = my^{m-1}x^n$ . Note that we have the following relations in  $\text{End}(\mathbb{C}[x, y])$ :  $\partial_y \partial_x = q \partial_x \partial_y$ ,  $\partial_x y = qy \partial_x$ ,  $\partial_y x = q^{-1}x \partial_y$ ,  $[\partial_x, x] = [\partial_y, y] = 1$ ,  $q^{2y \partial_y} \partial_y q^{-2y \partial_y} = q^{-2} \partial_y$ , where  $q^{\pm 2y \partial_y} = e^{\pm 2 \log(q)y \partial_y}$  are operators acting as  $q^{\pm 2y \partial_y}(x^n y^m) = q^{\pm 2m} x^n y^m$ . Then the matrix

$$M = \begin{pmatrix} x & q^{-1} q^{-2y \partial_y} \partial_y \\ y & q^{-2y \partial_y} \partial_x \end{pmatrix} \quad (2.13)$$

is a  $q$ -Manin matrix.

- Examples of  $q$ -Manin matrices related to the quantum groups  $\text{Fun}_q(GL_n)$  and  $U_q(\widehat{\mathfrak{gl}}_n)$  will be considered in Subsections 2.4 and 6.1 respectively.

**Remark 2.2.** If an  $n \times m$  matrix  $M$  is a  $q$ -Manin matrix, then the  $n \times m$  matrix  $\widetilde{M}$  with entries  $\widetilde{M}_{ij} = M_{n-i+1, m-j+1}$  is a  $q^{-1}$ -Manin matrix. For example, if  $M$  is the matrix (2.9) then we obtain the following  $q^{-1}$ -Manin matrix:

$$\widetilde{M} = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \quad (2.14)$$

Let us note that matrices obtained by permutations of rows or columns from  $q$ -Manin matrices are *not*  $q$ -Manin matrices in general.

## 2.2 Linear transformations of $q$ -(anti)-commuting variables

The original definition of a Manin matrix [30], for the case  $q = 1$ , was that of a matrix defining a linear transformation of commuting variables – generators of the polynomial algebra – as well as for a linear transformation of anti-commuting variables – generators of the Grassmann algebra. This was largely used in [4, 5]. Let us consider here an analogous interpretation of the  $q$ -Manin matrices.



Let us first introduce a  $q$ -deformation of the polynomial algebra and a  $q$ -deformation of the Grassmann algebra. Define the  $q$ -polynomial algebra  $\mathbb{C}[x_1, \dots, x_m]$  as the algebra with generators  $x_i$ ,  $i = 1, \dots, m$ , and relations

$$x_i x_j = q^{-1} x_j x_i$$

for  $i < j$ . These relations can be rewritten in the form

$$x_i x_j = q^{\text{sgn}(i-j)} x_j x_i, \quad i, j = 1, \dots, m. \quad (2.15)$$

The elements  $x_i$  are called  $q$ -polynomial variables. Similarly define the  $q$ -Grassmann algebra  $\mathbb{C}[\psi_1, \dots, \psi_n]$  as the algebra generated by  $\psi_i$ ,  $i = 1, \dots, n$ , with relations  $\psi_i^2 = 0$  for  $i = 1, \dots, n$  and  $\psi_i \psi_j = -q \psi_j \psi_i$ , for  $i < j$ ; that is

$$\psi_i \psi_j = -q^{-\text{sgn}(i-j)} \psi_j \psi_i, \quad i, j = 1, \dots, n. \quad (2.16)$$

The elements  $\psi_i$  are called  $q$ -Grassmann variables.

Let  $M$  be a rectangular  $n \times m$ -matrix over  $\mathfrak{A}$ . We can always suppose that  $\mathfrak{A}$  contains the  $q$ -polynomial algebra  $\mathbb{C}[x_1, \dots, x_m]$  and the  $q$ -Grassmann algebra  $\mathbb{C}[\psi_1, \dots, \psi_n]$  as subalgebras such that the elements of these subalgebras commute with the entries of  $M$ :  $[x_j, M_{pq}] = [\psi_i, M_{kl}] = 0$  for all  $i, k = 1, \dots, n$  and  $j, l = 1, \dots, m$ . Indeed, if  $\mathfrak{A}$  does not contain one of these algebras we can regard the matrix  $M$  as a matrix over the algebra  $\mathfrak{A} \otimes \mathbb{C}[x_1, \dots, x_m] \otimes \mathbb{C}[\psi_1, \dots, \psi_n]$ , where the elements of different tensor factors pairwise commute.

**Proposition 2.1.** *Let the entries of a rectangular  $n \times m$  matrix  $M$  commute with the variables  $x_1, \dots, x_m$  and  $\psi_1, \dots, \psi_n$ . Consider new  $q$ -polynomial variables  $\tilde{x}_1, \dots, \tilde{x}_n \in \mathfrak{A}$  and new  $q$ -Grassmann variables  $\tilde{\psi}_1, \dots, \tilde{\psi}_m \in \mathfrak{A}$  obtained by left (right) ‘action’ of  $M$  on the old variables:*

$$\begin{pmatrix} \tilde{x}_1 \\ \dots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \dots & & \\ M_{n1} & \dots & M_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_m \end{pmatrix} \quad (2.17)$$

$$(\tilde{\psi}_1, \dots, \tilde{\psi}_m) = (\psi_1, \dots, \psi_n) \begin{pmatrix} M_{11} & \dots & M_{1m} \\ \dots & & \\ M_{n1} & \dots & M_{nm} \end{pmatrix}, \quad (2.18)$$

Then the following three conditions are equivalent:

- The matrix  $M$  is a  $q$ -Manin matrix.
- The variables  $\tilde{x}_i$   $q$ -commute:  $\tilde{x}_i \tilde{x}_j = q^{\text{sgn}(i-j)} \tilde{x}_j \tilde{x}_i$  for all  $i, j = 1, \dots, n$ .
- The variables  $\tilde{\psi}_i$   $q$ -anticommute:  $\tilde{\psi}_i \tilde{\psi}_j = -q^{-\text{sgn}(i-j)} \tilde{\psi}_j \tilde{\psi}_i$  for all  $i, j = 1, \dots, m$ .

**Example 2.1.** Let  $x_1, x_2, \psi_1, \psi_2 \in \mathfrak{A}$  be elements such that  $x_1 x_2 = q^{-1} x_2 x_1$ ,  $\psi_1 \psi_2 = -q \psi_2 \psi_1$ ,  $\psi_1^2 = \psi_2^2 = 0$ . In the case  $n = 2$  the formulae (2.17), (2.18) take the form

$$\tilde{x}_1 = ax_1 + bx_2, \quad \tilde{\psi}_1 = a\psi_1 + c\psi_2, \quad (2.19)$$

$$\tilde{x}_2 = cx_1 + dx_2, \quad \tilde{\psi}_2 = b\psi_1 + d\psi_2, \quad (2.20)$$

where  $a, b, c, d \in \mathfrak{R}$  are elements commuting with  $x_1, x_2$  and with  $\psi_1, \psi_2$ . The matrix  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is  $q$ -Manin if and only if  $\tilde{x}_1\tilde{x}_2 = q^{-1}\tilde{x}_2\tilde{x}_1$ , or if and only if  $\tilde{\psi}_1\tilde{\psi}_2 = -q\tilde{\psi}_2\tilde{\psi}_1$  and  $\tilde{\psi}_1^2 = \tilde{\psi}_2^2 = 0$ .

Both factors of the algebra  $\mathfrak{R} \otimes \mathbb{C}[x_1, \dots, x_n]$  have a natural grading, and so  $q$ -Manin matrices can be interpreted as matrices of grading-preserving homomorphisms  $\mathbb{C}[x_1, \dots, x_n] \rightarrow \mathfrak{R} \otimes \mathbb{C}[x_1, \dots, x_n]$  with respect to the variables  $x_i$ , and/or matrices of grading-preserving homomorphisms  $\mathbb{C}[\psi_1, \dots, \psi_n] \rightarrow \mathfrak{R} \otimes \mathbb{C}[\psi_1, \dots, \psi_n]$  with respect to the variables  $\psi_i$ .

**Remark 2.3.** The conditions  $\tilde{\psi}_i^2 = 0$  are equivalent to the relations (2.1) and the conditions  $\tilde{\psi}_i\tilde{\psi}_j = -q\tilde{\psi}_j\tilde{\psi}_i$ ,  $i < j$ , are equivalent to the relations (2.2).

## 2.3 Elementary properties

**Proposition 2.2.** *The following properties hold:*

1. Any submatrix of a  $q$ -Manin matrix is also a  $q$ -Manin matrix.
2. Any diagonal matrix with commuting entries is a  $q$ -Manin matrix.
3. If  $M$  and  $N$  are  $q$ -Manin matrices, then  $M + N$  is a  $q$ -Manin matrix if and only if

$$\begin{aligned} M_{ij}N_{kl} - q^{\text{sgn}(i-k)}q^{-\text{sgn}(j-l)}M_{kl}N_{ij} - q^{\text{sgn}(i-k)}M_{kj}N_{il} + q^{-\text{sgn}(j-l)}M_{il}N_{kj} + \\ + N_{ij}M_{kl} - q^{\text{sgn}(i-k)}q^{-\text{sgn}(j-l)}N_{kl}M_{ij} - q^{\text{sgn}(i-k)}N_{kj}M_{il} + q^{-\text{sgn}(j-l)}N_{il}M_{kj} = 0, \end{aligned} \quad (2.21)$$

for all  $i, j, k, l = 1, \dots, n$ ; in particular, if

$$M_{ij}N_{kl} = q^{\text{sgn}(i-k)}q^{-\text{sgn}(j-l)}N_{kl}M_{ij}, \quad (2.22)$$

for all  $i, j, k, l = 1, \dots, n$  then  $M + N$  is a  $q$ -Manin matrix;

4. The product  $cM$  of  $q$ -Manin matrix  $M$  and a complex constant  $c \in \mathbb{C}$  is also a  $q$ -Manin matrix.
5. The product of  $q$ -Manin matrix  $M$  and a diagonal complex matrix  $D$  from the left  $DM$  as well as from the right  $MD$  is also a  $q$ -Manin matrix.
6. Let  $M$  and  $N$  be  $n \times m$  and  $m \times r$   $q$ -Manin matrices over an algebra  $\mathfrak{R}$  such that their elements commute, i.e.  $[M_{ij}, N_{kl}] = 0$ ,  $i = 1, \dots, n$ ,  $j, k = 1, \dots, m$ ,  $l = 1, \dots, r$ , then the product  $MN$  is a  $q$ -Manin matrix.

**Proof.** The first two properties (1 and 2) are obvious. To prove property 3 one should write the relations (2.3) for  $M + N$ . Taking into account the relations (2.3) for  $M$  and  $N$  one arrives to the relation (2.21). The condition (2.22) implies the condition (2.21).

The properties 4 and 5 are particular cases of the property 6. To prove the last one we consider  $q$ -commuting variables<sup>1</sup>  $x_i$ ,  $i = 1, \dots, r$ , commuting with  $M$  and  $N$ :

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<sup>1</sup>One can just as well use  $q$ -Grassmann variables  $\psi_i$ .

$x_i x_j = q^{\text{sgn}(i-j)} x_j x_i$ ,  $[M_{ij}, x_l] = [N_{ij}, x_l] = 0$ . Due to Proposition 2.1 new variables  $x_k^N = \sum_{l=1}^r N_{kl} x_l$ ,  $k = 1, \dots, m$ ,  $q$ -commute:  $x_i^N x_j^N = q^{\text{sgn}(i-j)} x_j^N x_i^N$ . Since  $[M_{ij}, x_k^N] = 0$  we can apply Proposition 2.1 to the matrix  $M$  and the variables  $x_k^N$  – the variables  $x_i^{MN} = \sum_{j=1}^m M_{ij} x_j^N$ ,  $i = 1, \dots, n$ , also  $q$ -commute:  $x_i^{MN} x_j^{MN} = q^{\text{sgn}(i-j)} x_j^{MN} x_i^{MN}$ . Then, the formula  $x_i^{MN} = \sum_{l=1}^r (MN)_{il} x_l$  and Proposition 2.1 imply that  $MN$  is a  $q$ -Manin matrix.  $\square$

**Remark 2.4.** If  $M$  and  $N$  are matrices over the algebras  $\mathfrak{R}_M$  and  $\mathfrak{R}_N$  respectively we can consider  $M$  and  $N$  as matrices over the same algebra  $\mathfrak{R}_M \otimes \mathfrak{R}_N$  and we then have the condition  $[M_{ij}, N_{kl}] = 0$  for all  $i, j, k, l$ .

## 2.4 Relations with quantum groups

One can also define  $q$ -analogues of Manin matrices characterizing the connections to quantum group theory. Actually  $q$ -Manin matrices are defined by half of the relations of the corresponding quantum group  $\text{Fun}_q(GL_n)^2$  ([13]). The remaining half consists of relations insuring that  $M^\top$  is also a  $q$ -Manin matrix, where  $M^\top$  is the transpose of  $M$ .

**Definition 2.** An  $n \times n$  matrix  $T$  belongs to the quantum group  $\text{Fun}_q(GL_n)$  if the following conditions hold true. For any  $2 \times 2$  submatrix  $(T_{(ij)(kl)})$ , consisting of rows  $i$  and  $k$ , and columns  $j$  and  $l$  (where  $1 \leq i < k \leq n$ , and  $1 \leq j < l \leq n$ ):

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & T_{ij} & \dots & T_{il} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & T_{kj} & \dots & T_{kl} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \equiv \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & a & \dots & b & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & c & \dots & d & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (2.23)$$

the following commutation relations hold:

$$ca = qac, \quad (q\text{-commutation of the entries in a column}) \quad (2.24)$$

$$db = qbd, \quad (q\text{-commutation of the entries in a column}) \quad (2.25)$$

$$ba = qab, \quad (q\text{-commutation of the entries in a row}) \quad (2.26)$$

$$dc = qcd, \quad (q\text{-commutation of the entries in a row}) \quad (2.27)$$

$$ad - da = +q^{-1}cb - qbc, \quad (\text{cross commutation relation 1}) \quad (2.28)$$

$$bc = cb, \quad (\text{cross commutation relation 2}). \quad (2.29)$$

As quantum groups are usually defined within the so-called matrix (Leningrad) formalism, let us briefly recall it. (We will further discuss this issue in Section 5).

**Lemma 2.3.** The commutation relations for quantum group matrices can be described in matrix (Leningrad) notations as follows:

$$R(T \otimes 1)(1 \otimes T) = (1 \otimes T)(T \otimes 1)R, \quad (2.30)$$

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<sup>2</sup>More precisely we should write  $\text{Fun}_q(\text{Mat}_n)$ , since we do not localize the  $q$ -determinant.

The  $R$ -matrix is given, in the case we are considering, by the formula:

$$R = q^{-1} \sum_{i=1,\dots,n} E_{ii} \otimes E_{ii} + \sum_{i,j=1,\dots,n; i \neq j} E_{ii} \otimes E_{jj} + (q^{-1} - q) \sum_{i,j=1,\dots,n; i > j} E_{ij} \otimes E_{ji}, \quad (2.31)$$

where  $E_{ij}$  are the standard basis of  $\text{End}(\mathbb{C}^n)$ , i.e.  $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$  - zeroes everywhere except 1 in the intersection of the  $i$ -th row with the  $j$ -th column.

For example in the  $2 \times 2$  case the  $R$ -matrix is:

$$R = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}. \quad (2.32)$$

**Remark 2.5.** This  $R$ -matrix differs by the change  $q \rightarrow q^{-1}$  from that one in [13], page 185.

The relation between  $q$ -Manin matrices and quantum groups consists in the following simple proposition:

**Proposition 2.4.** *A matrix  $T$  is a matrix in the quantum group  $\text{Fun}_q(GL_n)$  if and only if both  $T$  and  $T^\top$  are  $q$ -Manin matrices.*

So one gets that  $q$ -Manin matrices are characterized by a "half" of the conditions of the corresponding quantum matrix group.

## 2.5 Hopf structure

Let us consider the algebra  $\mathbb{C}[M_{ij}]$  generated (over  $\mathbb{C}$ ) by  $M_{ij}$ ,  $1 \leq i, j \leq n$ , with  $q$ -Manin relations (2.3). One can see that it can be equipped by a structure of bialgebra with the coproduct  $\Delta(M_{ij}) = \sum_k M_{ik} \otimes M_{kj}$ . This is usually denoted as follows:

$$\Delta(M) = M \dot{\otimes} M. \quad (2.33)$$

It is easy to see that this coproduct is coassociative (i.e.  $(\Delta \otimes 1) \otimes \Delta = (1 \otimes \Delta) \otimes \Delta$ ).

The natural antipode for this bialgebra should be  $S(M) = M^{-1}$ . So it exists only in some extensions of the algebra  $\mathbb{C}[M_{ij}]$ .

The "coaction"-proposition 2.1 page 8 implies that there exist morphisms of algebras:

$$\phi_1 : \mathbb{C}[x_1, \dots, x_m] \rightarrow \mathbb{C}[M_{ij}] \otimes \mathbb{C}[x_1, \dots, x_m], \quad \phi_1(x_i) = \sum_k M_{ik} x_k, \quad (2.34)$$

$$\phi_2 : \mathbb{C}[\psi_1, \dots, \psi_n] \rightarrow \mathbb{C}[M_{ij}] \otimes \mathbb{C}[\psi_1, \dots, \psi_n], \quad \phi_2(\psi_i) = \sum_k M_{ki} \psi_k. \quad (2.35)$$

One can check that both maps satisfy the condition:  $(\Delta \otimes 1)(\phi_i) = (1 \otimes \phi_i)(\phi_i)$ ,  $i = 1, 2$ .

So one can consider the maps  $\phi_i$  as "coactions" of  $q$ -Manin matrices on the space  $\mathbb{C}_q^n$  and its Grassmannian version.

### 3 The $q$ -determinant and Cramer's formula

It was shown in [4, 5] that the natural generalization of the usual determinant for Manin matrices (i.e., the  $q = 1$  case) is the column determinant. This column determinant satisfies all the properties of the determinant of square matrices over a commuting field, and is defined as in the usual case, with the proviso in mind that the orders of the column index in the  $n!$  summands of the determinant should be always the same. The role of column determinant for  $q$ -Manin matrices is played by its  $q$ -analogue called  $q$ -determinant. Most of the properties of column determinant of Manin matrices presented in [5] can be generalized to general  $q$ .

In this section we recall the definition of the  $q$ -determinant. We will see that in the case of  $q$ -Manin matrices it generalizes the notion of the usual determinants for the matrices over commutative rings. We shall start by considering the  $q$ -determinant for an arbitrary (i.e., not necessarily  $q$ -Manin) matrix with elements in a noncommutative ring  $\mathfrak{R}$ .

#### 3.1 The $q$ -determinant

We define the  $q$ -determinant of an arbitrary matrix  $M$  with non-commutative entries and relate it with coaction of  $M$  on the  $q$ -Grassmann algebra. We prove here some formulae used below for  $q$ -determinants and  $q$ -minors of  $q$ -Manin matrices.

##### 3.1.1 Definition of the $q$ -determinant

**Definition 3.** The  $q$ -determinant, of a  $n \times n$  matrix  $M = (M_{ij})$  is defined by the formula

$$\det_q M := \sum_{\tau \in \mathfrak{S}_n} (-q)^{-\text{inv}(\tau)} M_{\tau(1)1} M_{\tau(2)2} \cdots M_{\tau(n)n}, \quad (3.1)$$

where the sum ranges over the set  $\mathfrak{S}_n$  of all permutations of  $\{1, \dots, n\}$ . Recall that  $\text{inv}(\tau)$  is number of inversions – the number of pairs  $1 \leq i < j \leq n$  for which  $\tau(i) > \tau(j)$ . In other words,  $\text{inv}(\tau)$  is the length of  $\tau$  with respect to adjacent transpositions  $\sigma_k = \sigma_{(k, k+1)}$ .

**Example 3.1.**

$$\det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\text{def}}{=} ad - q^{-1}cb. \quad (3.2)$$

**Example 3.2.**

$$\det_q \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \stackrel{\text{def}}{=} M_{11}M_{22}M_{33} + q^{-2}M_{21}M_{32}M_{13} + q^{-2}M_{31}M_{12}M_{23} -$$

$$-q^{-1}M_{11}M_{32}M_{23} - q^{-1}M_{21}M_{12}M_{33} - q^{-3}M_{31}M_{22}M_{13} = \quad (3.3)$$

$$= M_{11} \det_q \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} + (-q)^{-1} M_{21} \det_q \begin{pmatrix} M_{12} & M_{13} \\ M_{32} & M_{33} \end{pmatrix} +$$

$$(-q)^{-2} M_{31} \det_q \begin{pmatrix} M_{12} & M_{13} \\ M_{22} & M_{23} \end{pmatrix} = (-q)^{-2} \det_q \begin{pmatrix} M_{21} & M_{22} \\ M_{31} & M_{32} \end{pmatrix} M_{13} +$$

$$(-q)^{-1} \det_q \begin{pmatrix} M_{11} & M_{12} \\ M_{31} & M_{32} \end{pmatrix} M_{23} + \det_q \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} M_{33}. \quad (3.4)$$

It is easy to see from the definition that, if  $L$  is an arbitrary lower-triangular matrix with  $d_i$  on the diagonal, and  $R$  an arbitrary upper-triangular matrix with  $d_i$  on the diagonal, then

$$\det_q L = \det_q R = \prod_i d_i. \quad (3.5)$$

It is also easy to see that, if,  $S$  is the permutation matrix corresponding to the permutation  $\sigma \in \mathfrak{S}_n$ , then  $\det_q S = (-q)^{-\text{inv}(\sigma)}$ .

**Remark 3.1.** For  $q = 1$  the present definition of  $q$ -determinant coincides with that of column-determinant i.e. the determinant defined by the column expansion, first taking elements from the first column, then from the second and so on and so forth. For  $q = -1$  the  $q$ -determinant yields the (column) permanent of the matrix  $M$ . Just like the usual determinant, the  $q$ -determinant is linear over  $\mathbb{C}$  both with respect to its columns and rows.

### 3.1.2 The $q$ -Grassmann algebra

Let us collect some useful relations in the  $q$ -Grassmann algebra  $\mathbb{C}[\psi_1, \dots, \psi_n]$ . Some of them hold without assuming  $\psi_i^2 = 0$ , while some of them require this property.

**Lemma 3.1.** *Let  $\psi_i$  satisfy the relations  $\psi_i \psi_j = -q \psi_j \psi_i$  for all  $1 \leq i < j \leq n$ ; then their monomials of  $n$ -th order are related as follows*

$$\psi_{\tau(1)} \cdots \psi_{\tau(n)} = (-q)^{-\text{inv}(\tau)} \psi_1 \cdots \psi_n, \quad \forall \tau \in \mathfrak{S}_n, \quad (3.6)$$

or equivalently as

$$\psi_{\sigma\tau(1)} \cdots \psi_{\sigma\tau(n)} = (-q)^{-\text{inv}(\sigma\tau) + \text{inv}(\sigma)} \psi_{\sigma(1)} \cdots \psi_{\sigma(n)}, \quad \forall \sigma, \tau \in \mathfrak{S}_n. \quad (3.7)$$

**Proof.** Note that if the relation (3.7) is valid for some  $\tau_1, \tau_2 \in \mathfrak{S}_n$  (for all  $\sigma \in \mathfrak{S}_n$ ) then it is so for  $\tau = \tau_1 \tau_2$ . Indeed

$$\begin{aligned} \psi_{\sigma\tau_1\tau_2(1)} \cdots \psi_{\sigma\tau_1\tau_2(n)} &= (-q)^{-\text{inv}(\sigma\tau_1\tau_2) + \text{inv}(\sigma\tau_1)} \psi_{\sigma\tau_1(1)} \cdots \psi_{\sigma\tau_1(n)} = \\ &= (-q)^{-\text{inv}(\sigma\tau_1\tau_2) + \text{inv}(\sigma)} \psi_{\sigma(1)} \cdots \psi_{\sigma(n)}. \end{aligned} \quad (3.8)$$

Since each  $\tau$  can be presented as a product of adjacent transpositions  $\sigma_k$ ,  $k = 1, \dots, n-1$ , it is sufficient to prove (3.7) for  $\tau = \sigma_k$ . In this case we can write

$$\begin{aligned} \psi_{\sigma\sigma_k(1)} \cdots \psi_{\sigma\sigma_k(n)} &= \psi_{\sigma(1)} \cdots \psi_{\sigma(k+1)} \psi_{\sigma(k)} \cdots \psi_{\sigma(n)} = \\ &= (-q)^{-\text{sgn}(\sigma(k+1) - \sigma(k))} \psi_{\sigma(1)} \cdots \psi_{\sigma(k)} \psi_{\sigma(k+1)} \cdots \psi_{\sigma(n)}. \end{aligned} \quad (3.9)$$

Thus, formula (3.7) for  $\tau = \sigma_k$  follows from the equality

$$\text{inv}(\sigma\sigma_k) = \text{inv}(\sigma) + \text{sgn}(\sigma(k+1) - \sigma(k)). \quad (3.10)$$

□

**Corollary 3.1.1.** *Under the condition of the Lemma 3.1, one can relate the monomials of  $m$ -th order as follows:*

$$\psi_{j_{\tau(1)}} \cdots \psi_{j_{\tau(m)}} = (-q)^{-\text{inv}(\tau)} \psi_{j_1} \cdots \psi_{j_m}, \quad (3.11)$$

$$\psi_{j_{\tau(1)}} \cdots \psi_{j_{\tau(m)}} = (-q)^{-\text{inv}(\tau) + \text{inv}(\sigma)} \psi_{j_{\sigma(1)}} \cdots \psi_{j_{\sigma(m)}}, \quad (3.12)$$

where  $1 \leq j_1 < \dots < j_m \leq n$ ,  $\tau, \sigma \in \mathfrak{S}_m$ .

**Proof.** Consider the elements  $\tilde{\psi}_k = \psi_{j_k}$ . They satisfy the conditions of Lemma 3.1:  $\tilde{\psi}_i \tilde{\psi}_j = -q \tilde{\psi}_j \tilde{\psi}_i$  for all  $1 \leq i < j \leq m$ . Writing the formulae (3.6), (3.7) for  $\tilde{\psi}_k$  we obtain (3.11), (3.12). □

**Corollary 3.1.2.** *Assuming additionally  $\psi_i^2 = 0$ , i.e.  $\psi_i \psi_j = -q^{-\text{sgn}(i-j)} \psi_j \psi_i$ , it is convenient to write the more general formula*

$$\psi_{i_1} \cdots \psi_{i_n} = \varepsilon_{i_1, \dots, i_n}^q \psi_1 \cdots \psi_n, \quad (3.13)$$

where  $1 \leq i_l \leq n$  and the  $q$ -epsilon-symbol is defined by the formula

$$\varepsilon_{\dots, i, \dots, i, \dots}^q = 0, \quad \varepsilon_{\sigma(1), \dots, \sigma(n)}^q = (-q)^{-\text{inv}(\sigma)}, \quad \sigma \in \mathfrak{S}_n. \quad (3.14)$$

Let us denote by  $I \oplus J = (i_1, \dots, i_m, j_1, \dots, j_k)$  the contraction of two multi-indices  $I = (i_1, \dots, i_m)$  and  $J = (j_1, \dots, j_k)$ . For a multi-index  $K = (k_1, k_2, \dots, k_r)$  denote by  $\backslash K$  the multi-index  $(1, \dots, \hat{k}_1, \dots, \hat{k}_m, \dots, n)$ , that is  $\backslash K$  is obtained from  $(1, 2, \dots, n)$  by deleting  $k_i$  for all  $i = 1, \dots, r$ .

Let  $(i_1, \dots, i_n)$  be a permutation of  $(1, \dots, n)$  such that  $i_1 < i_2 < \dots < i_m$  and  $i_{m+1} < i_{m+2} < \dots < i_n$  for some  $m \leq n$ . Let  $I = (i_1, i_2, \dots, i_m)$  and  $\backslash I = (i_{m+1}, i_{m+2}, \dots, i_n)$ . It is quite easy to see that

$$\varepsilon_{(I \oplus \backslash I)}^q = (-q)^{-\sum_{l=1}^m (i_l - l)} = (-q)^{+\sum_{l=m+1}^n (i_l - l)}, \quad (3.15)$$

$$\varepsilon_{(\backslash I \oplus I)}^q = (-q)^{\sum_{l=1}^m i_l - \sum_{l=n-m+1}^n l} = (-q)^{\sum_{l=1}^{n-m} l - \sum_{l=m+1}^n i_l}, \quad (3.16)$$



where we used  $\sum_{l=1,\dots,n} i_l = \sum_{l=1,\dots,n} l$ .

Let us also mention that for  $I = (n, n-1, n-2, \dots, 1)$  we have

$$\varepsilon_I^q = (-q)^{-n(n-1)/2}. \quad (3.17)$$

### 3.1.3 The $q$ -determinant, $q$ -minors and the $q$ -Grassmann algebra

Let us give a more conceptual approach to  $q$ -determinants and  $q$ -minors –  $q$ -determinants of submatrices – via the  $q$ -Grassmann algebra. We consider an arbitrary matrix  $M$ , but the fact that the notion of the  $q$ -determinant can be reformulated in terms of the  $q$ -Grassmann algebra is a clear hint that it is related with the notion of  $q$ -Manin matrix.

Let  $M$  be an arbitrary  $n \times m$  matrix and  $I = (i_1, i_2, \dots, i_k)$  and  $J = (j_1, j_2, \dots, j_l)$  be two arbitrary multi-indices, where  $k \leq n$  and  $l \leq m$ . Denote by  $M_{IJ}$  the  $k \times l$  matrix defined as  $(M_{IJ})_{ab} = M_{i_a j_b}$ , where  $a, b = 1, \dots, k$ . Then the  $q$ -determinant  $\det_q(M_{IJ})$  can be considered as the  $q$ -analogue of the minor ( $q$ -minor) of the matrix  $M$ .

**Proposition 3.2.** *Let  $M$  be an arbitrary (not-necessarily  $q$ -Manin)  $n \times n$  matrix,  $\psi_i$  –  $q$ -Grassmann variables, i.e.  $\psi_i \psi_j = -q^{-\text{sgn}(i-j)} \psi_j \psi_i$ , which commute with the entries of  $M$ , and  $\psi_i^M = \sum_k \psi_k M_{ki}$ . Then*

$$\psi_1^M \psi_2^M \dots \psi_n^M = \det_q(M) \psi_1 \psi_2 \dots \psi_n. \quad (3.18)$$

More generally, for a rectangular  $n \times m$  matrix  $M$  and for an arbitrary<sup>3</sup> multi-index  $J = (j_1, j_2, \dots, j_k)$

$$\psi_{j_1}^M \psi_{j_2}^M \dots \psi_{j_k}^M = \sum_{I=(i_1 < i_2 < \dots < i_k)} \det_q(M_{IJ}) \psi_{i_1} \psi_{i_2} \dots \psi_{i_k}, \quad (3.19)$$

where the sum is taken over all multi-indices  $I = (i_1, i_2, \dots, i_k)$  such that  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ .

**Proof.** The formula (3.18) follows from (3.19) in the case  $m = n = k$ . To prove the last one we write left hand side as

$$\psi_{j_1}^M \dots \psi_{j_k}^M = \sum_{l_1, \dots, l_k=1}^n M_{l_1 j_1} \dots M_{l_k j_k} \psi_{l_1} \dots \psi_{l_k}. \quad (3.20)$$

The product  $\psi_{l_1} \dots \psi_{l_k}$  does not vanish only if  $l_1, \dots, l_k$  is a permutation of some numbers  $i_1, \dots, i_k$  such that  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ . Then using the formula (3.11) one derives:

$$\begin{aligned} \psi_{j_1}^M \dots \psi_{j_k}^M &= \sum_{I=(i_1 < i_2 < \dots < i_k)} \sum_{\tau \in \mathfrak{S}_k} M_{i_{\tau(1)} j_1} \dots M_{i_{\tau(k)} j_k} \psi_{i_{\tau(1)}} \dots \psi_{i_{\tau(k)}} = \\ &= \sum_{I=(i_1 < i_2 < \dots < i_k)} \sum_{\tau \in \mathfrak{S}_k} (-q)^{-\text{inv}(\tau)} M_{i_{\tau(1)} j_1} \dots M_{i_{\tau(k)} j_k} \psi_{i_1} \dots \psi_{i_k}. \end{aligned} \quad (3.21)$$

---

<sup>3</sup>Here it is *neither* required that  $j_l < j_p$ , nor that  $j_l \neq j_p$ .

On the other hand the  $q$ -determinant of  $M_{IJ}$  can be written as

$$\det_q(M_{IJ}) = \sum_{\tau \in \mathfrak{S}_k} (-q)^{-\text{inv}(\tau)} M_{i_{\tau(1)}, j_1} \cdots M_{i_{\tau(n)}, j_k}. \quad (3.22)$$

Comparing (3.21) with (3.22) one gets (3.19).  $\square$

**Example 3.3.** For  $\psi_1\psi_2 = -q\psi_2\psi_1$ ,  $\psi_i^2 = 0$  and  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , one has

$$\begin{aligned} \psi_1^M \psi_2^M &= (\psi_1 a + \psi_2 c)(\psi_1 b + \psi_2 d) = \psi_1 \psi_2 a d + \psi_2 \psi_1 c b = \\ &= \psi_1 \psi_2 a d - q^{-1} \psi_1 \psi_2 c b = \det_q(M) \psi_1 \psi_2. \end{aligned} \quad (3.23)$$

A tautological corollary of the proposition above is the following.

**Corollary 3.2.1.** *Let  $M^\sigma$  be the matrix obtained from  $M$  by the permutation  $\sigma \in \mathfrak{S}_n$  of columns:  $M_{ij}^\sigma = M_{i\sigma(j)}$ . Then*

$$\psi_{\sigma(1)}^M \psi_{\sigma(2)}^M \cdots \psi_{\sigma(n)}^M = \det_q(M^\sigma) \psi_1 \psi_2 \cdots \psi_n. \quad (3.24)$$

In Proposition 3.2 it is crucial that the elements of  $M$  commute with  $\psi_i$ ; however  $q$ -Grassmann algebra is also helpful without this condition. This can be seen from the following formula which is proved in the same way as Proposition 3.2

$$\sum_{\substack{I=(i_1, \dots, i_r) \\ 1 \leq i_a \leq n}} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_r} M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_r j_r} = \sum_{L=(l_1 < l_2 < \dots < l_r)} \psi_{l_1} \psi_{l_2} \cdots \psi_{l_r} \det_q(M_{LJ}).$$

The Laplace expansion can also be written as follows:

$$\begin{aligned} \sum_{L_1=(l_1 < l_2 < \dots < l_a)} \psi_{l_1} \cdots \psi_{l_a} \sum_{L_2=(l_{a+1} < l_{a+2} < \dots < l_r)} \psi_{l_{a+1}} \cdots \psi_{l_r} \det_q A_{L_1(j_1, \dots, j_a)} \times \\ \times \det_q(A_{L_2(j_{a+1}, \dots, j_r)}) = \sum_{L=(l_1 < l_2 < \dots < l_r)} \psi_{l_1} \psi_{l_2} \cdots \psi_{l_r} \det_q(A_{LJ}). \end{aligned} \quad (3.25)$$

## 3.2 Properties of the $q$ -determinants of $q$ -Manin matrices

The results of the previous subsection lead to the properties of  $q$ -determinants listed below. Proposition 3.2 allows us to derive some properties for  $q$ -determinants of  $q$ -Manin matrix. Let us remark that some these properties are valid for matrices with entries satisfying a part of the  $q$ -Manin relations or even for arbitrary matrices.

**Proposition 3.3.** *The following properties hold:*

1. Linearity in columns and rows.

*If some column (row) of a square non-commutative (not necessarily  $q$ -Manin) matrix  $M$  is presented as a sum of column-matrices (rows-matrices) then its  $q$ -determinant  $\det_q M$  is equal to the sum of  $q$ -determinants of the matrices  $M$  with the considered column (row) replaced by the corresponding column-matrices (rows-matrices).*

2. Permutation of columns.

If  $M$  is a square matrix satisfying relations (2.2) (in particular, if  $M$  is a  $q$ -Manin matrix) and  $M^\sigma$  denotes the matrix obtained from  $M$  by a permutation of columns:  $M_{ij}^\sigma = M_{i\sigma(j)}$ , where  $\sigma \in \mathfrak{S}_n$ , then

$$\det_q(M^\sigma) = (-q)^{-\text{inv}(\sigma)} \det_q M. \quad (3.26)$$

Note that  $M^\sigma$  is not a  $q$ -Manin matrix in general (even if  $M$  is a  $q$ -Manin matrix). Let us also remark that permutation of rows affects the  $q$ -determinant in a highly non-trivial way since the shuffling of the rows destroys the property of being  $q$ -Manin.

3. Matrices with coincident columns.

Let  $M$  be a square  $q$ -Manin matrix. If two columns of  $M$  coincide, then

$$\det_q M = 0. \quad (3.27)$$

If  $q \neq -1$  the same holds for any square  $M$  satisfying relations (2.2).

Furthermore, if  $M$  is a square  $q$ -Manin matrix and the matrix  $\widetilde{M}$  is obtained from  $M$  by substituting the  $r$ -th column to the  $s$ -th column, where  $s \neq r$ , then

$$\det_q \widetilde{M} = 0, \quad (3.28)$$

note that if  $|s - r| \geq 2$  then  $\widetilde{M}$  is not a  $q$ -Manin matrix in general.<sup>4</sup>

In particular, the coincidence of rows do not imply the vanishing of the  $q$ -determinant.

4. Determinant multiplicativity and Cauchy-Binet formula.

Let  $M$  and  $N$  be two matrices such that  $[M_{ij}, N_{kl}] = 0$  for all possible indices  $i, j, k, l$  and let  $M$  be a  $q$ -Manin matrix. If these are  $n \times n$  matrices, then

$$\det_q(MN) = \det_q(M) \det_q(N). \quad (3.29)$$

More generally, if  $M$  and  $N$  are rectangular matrices and  $i_1 < i_2 < \dots < i_r$  then the Cauchy-Binet formula holds:

$$\det_q((MN)_{IJ}) = \sum_{L=(l_1 < l_2 < \dots < l_r)} \det_q(M_{IL}) \det_q(N_{LJ}), \quad (3.30)$$

where  $I = (i_1, i_2, \dots, i_r)$  and  $J = (j_1, j_2, \dots, j_r)$ .

Recall that in this case  $MN$  is a  $q$ -Manin matrix, if  $N$  is also a  $q$ -Manin matrix (see the item 6 of Proposition 2.2 page 9).

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<sup>4</sup>Actually this property holds under weaker conditions: relations (2.2) and  $q$ -commutativity of elements in the  $r$ -th column.

5. Column expansion.

For an arbitrary  $n \times n$  matrix  $M$  (not necessarily a  $q$ -Manin matrix) the following expansions with respect to the first and last columns hold:

$$\det_q(M) = \sum_{r=1}^n (-q)^{1-r} M_{r1} \det_q(M_{\setminus r \setminus 1}) = \sum_{r=1}^n (-q)^{r-n} \det_q(M_{\setminus r \setminus n}) M_{rn}, \quad (3.31)$$

where  $M_{\setminus r \setminus s}$  is the  $(n-1) \times (n-1)$  matrix obtained by deleting the  $r$ -th row and the  $s$ -th column. (As an example see formulae (3.4)).

For an  $n \times n$  matrix  $M$  satisfying relations (2.2) (in particular, if  $M$  is a  $q$ -Manin matrix) its  $q$ -determinant can be decomposed (for any  $s = 1, \dots, n$ ) along the  $s$ -column as follows:

$$\det_q(M) = \sum_{r=1}^n (-q)^{s-r} M_{rs} \det_q(M_{\setminus r \setminus s}) = \sum_{r=1}^n (-q)^{r-s} \det_q(M_{\setminus r \setminus s}) M_{rs}. \quad (3.32)$$

To the best of our knowledge, there are no analogous formulae for the row expansion.

6. Laplace expansion.

For an arbitrary  $n \times n$  matrix  $M$  (not necessarily  $q$ -Manin matrix) the following is true:

$$\det_q M = \sum_{\substack{K=(k_1, \dots, k_m) \\ 1 \leq k_1 < \dots < k_m \leq n}} (-q)^{-\sum_{l=1}^m (k_l - l)} \det_q M_{K, (1 \dots m)} \det_q M_{\setminus K, (m+1, \dots, n)}. \quad (3.33)$$

For an  $n \times n$   $q$ -Manin matrix  $M$ , and arbitrary pair of multi-indices  $I_1 = (i_1, \dots, i_m)$  and  $I_2 = (i_{m+1}, \dots, i_n)$  one can also write:

$$\varepsilon_{i_1, \dots, i_n}^q \det_q M = \sum_{K=(k_1 < \dots < k_m)} (-q)^{-\sum_{l=1}^m (k_l - l)} \det_q M_{K, I_1} \det_q M_{\setminus K, I_2}, \quad (3.34)$$

where the  $q$ -epsilon-symbol is defined in (3.14).

One can also write similar formulae with products of more determinants in the right hand side. Consider  $r$  non-negative numbers  $l_1, \dots, l_r$  such that  $l_1 + l_2 + \dots + l_r = n$  and arbitrary multi-indices  $I_j = (i_{j1}, i_{j2}, \dots, i_{j, l_j})$ . Let  $(i_1, \dots, i_n) = I_1 \oplus I_2 \oplus \dots \oplus I_r$ , then

$$\varepsilon_{i_1, \dots, i_n}^q \det_q M = \sum_{K_1, \dots, K_r} \varepsilon_{k_1, \dots, k_n}^q \prod_{j=1}^r \det_q M_{K_j I_j}, \quad (3.35)$$

where the sum is taken over all multi-indices  $K_j = (k_{j1}, k_{j2}, \dots, k_{j, l_j})$  such that  $k_{j1} < k_{j2} < \dots < k_{j, l_j}$  and  $(k_1, \dots, k_n) = K_1 \oplus K_2 \oplus \dots \oplus K_r$ .

If  $(i_1, \dots, i_n) = (1, 2, 3, \dots, n)$  then the formula (3.35) holds for arbitrary square matrix. For  $l_1 = l_2 = \dots = l_n = 1$  this formula becomes the definition of the  $q$ -determinant.

7. Non centrality of  $q$ -determinant .

We remark that  $\det_q(M)$  (contrary to the case of Quantum Group theory) is not a central element in the algebra  $\mathbb{C}[M_{ij}]$  defined in the subsection 2.5; moreover, in general  $[\det_q(M), \text{tr}(M)] \neq 0$ . This difference between  $q$ -Manin matrices and quantum matrices can be traced back to the fact that the commutation relations between elements of a  $q$ -Manin matrix are more general than those that the elements of a "quantum matrix" do satisfy.

**Proof of Property 1.** Let us show the linearity in the first column and in the first row. The first one is a direct sequence of Definition 3: let  $M_{ij} = N_{ij}$  for  $j \neq 1$  then

$$\det_q(M + N) = \sum_{\tau \in \mathfrak{S}_n} (-q)^{-\text{inv}(\tau)} (M_{\tau(1)1} + N_{\tau(1)1}) M_{\tau(2)2} \cdots M_{\tau(n)n} = \det_q(M) + \det_q(N).$$

Analogously, if  $M_{ij} = N_{ij}$  for  $i \neq 1$ ,

$$\begin{aligned} \det_q(M + N) &= \sum_{\tau \in \mathfrak{S}_n} (-q)^{-\text{inv}(\tau)} M_{\tau(1)1} M_{\tau(2)2} \cdots (M_{1\tau^{-1}(1)} + N_{1\tau^{-1}(1)}) \cdots M_{\tau(n)n} \\ &= \det_q(M) + \det_q(N). \end{aligned}$$

□

**Proof of Property 2.** Let  $\psi_i$  be  $q$ -Grassmann variables commuting with  $M$ , that is,  $\psi_i \psi_j = -q^{-\text{sgn}(i-j)} \psi_j \psi_i$ ,  $[\psi_i, M_{jk}] = 0$ , and  $\psi_i^M = \sum_{k=1}^n \psi_k M_{ki}$ . From Remark 2.3 we conclude that  $\psi_i^M$  satisfy the condition of the lemma 3.1. Applying the formula (3.6) to the left hand side of (3.24) we obtain (3.26). □

**Proof of Property 3.** Let the matrix  $M$  satisfy (2.2). Let suppose also that its  $k$ -th and  $l$ -th columns coincide. If  $\psi_j^M = \sum_{i=1}^n \psi_i M_{ij}$  then  $\psi_k^M = \psi_l^M$  and  $M = M^{\sigma_{kl}}$ , where  $\sigma_{kl}$  is a permutation interchanging  $k$  and  $l$ . If  $M$  is  $q$ -Manin the variables  $\psi_j^M$  are  $q$ -Grassmann and hence the equality (3.18) implies  $\det_q(M) = 0$ . If  $q \neq -1$  the formula  $\det_q(M) = 0$  is a consequence of the lemma 3.3 for  $\sigma = \sigma_{kl}$ . Let  $\widetilde{\psi_j^M} = \sum_{i=1}^n \psi_i M_{ij}$ , then  $\widetilde{\psi_i^M} \widetilde{\psi_r^M} = -q^{-\text{sgn}(i-r)} \widetilde{\psi_r^M} \widetilde{\psi_i^M}$  and  $\widetilde{\psi_r^M} = \widetilde{\psi_s^M}$ . It is sufficient to obtain the formula  $\det_q(\widetilde{M}) = 0$  using the equality (3.18). □

**Proof of Property 4.** Let  $\psi_i$ ,  $i = 1, \dots, n$ , be  $q$ -Grassmann variables, which commute with the entries of  $M$  and  $N$ . Since the matrix  $M$  is  $q$ -Manin the variables  $\psi_l^M = \sum_{i=1}^n \psi_i M_{il}$  are also  $q$ -Grassmann. Let  $\psi_j^{MN} = \sum_{l=1}^m \psi_l^M N_{lj} = \sum_{i=1}^n \psi_i (MN)_{ij}$ . Note that  $\psi_i$ ,  $\psi_l^M$  and  $\psi_i^{MN}$  commute with  $M$ ,  $N$ ,  $MN$  respectively. So that we can write the formula (3.19) for these matrices:

$$\psi_{l_1}^M \cdots \psi_{l_r}^M = \sum_{I=(i_1 < \dots < i_r)} \psi_{i_1} \cdots \psi_{i_r} \det_q(M_{IL}), \quad (3.36)$$

$$\psi_{j_1}^{MN} \cdots \psi_{j_r}^{MN} = \sum_{L=(l_1 < \dots < l_r)} \psi_{l_1}^M \cdots \psi_{l_r}^M \det_q(N_{LJ}), \quad (3.37)$$

$$\psi_{j_1}^{MN} \cdots \psi_{j_r}^{MN} = \sum_{I=(i_1 < \dots < i_r)} \psi_{i_1} \cdots \psi_{i_r} \det_q((MN)_{IJ}). \quad (3.38)$$

where  $L = (l_1 < \dots < l_r)$  and  $J = (j_1 < \dots < j_r)$ . Substituting (3.36) to the right hand side of (3.37) and comparing the result with (3.38) we derive (3.30).  $\square$

**Proof of Property 5.** The formula (3.31) follows immediately from the definition of the  $q$ -determinant. For example,

$$\begin{aligned} \det_q(M) &= \sum_{\tau \in \mathfrak{S}_n} (-q)^{-\text{inv}(\tau)} M_{\tau(1),1} \cdots M_{\tau(n),n} = \\ &= \sum_{r=1}^n \sum_{\substack{\tau \in \mathfrak{S}_n \\ \tau(n)=r}} (-q)^{-\text{inv}(\tau)} M_{\tau(1),1} \cdots M_{\tau(n-1),n-1} M_{rn} = \sum_{r=1}^n (-q)^{r-n} \det_q(M_{\setminus r \setminus n}) M_{rn}, \end{aligned}$$

where we used the fact that if  $\tau(n) = r$  then  $(\tau(1), \dots, \tau(n-1))$  is a permutation of  $(1, \dots, r-1, r+1, \dots, n)$  of length  $\text{inv}(\tau) - (n-r)$ . The expansion (3.31) with respect to the first column can be shown in the same way.

To prove the formula (3.32) we consider the permutation  $\sigma = \begin{pmatrix} 1, \dots, s-1, s, s+1, \dots, n-1, n \\ 1, \dots, s-1, s+1, s+2, \dots, n, s \end{pmatrix}$ . Taking into account  $(M^\sigma)_{\setminus r \setminus n} = M_{\setminus r \setminus s}$ ,  $\text{inv}(\sigma) = n-s$  and the formula (3.26) one gets

$$\begin{aligned} \det_q(M) &= (-q)^{\text{inv}(\sigma)} \det_q(M^\sigma) = (-q)^{\text{inv}(\sigma)} \sum_{r=1}^n (-q)^{r-n} \det_q((M^\sigma)_{\setminus r \setminus n}) M_{rs} = \\ &= \sum_{r=1}^n (-q)^{r-s} \det_q(M_{\setminus r \setminus s}) M_{rs}. \quad (3.39) \end{aligned}$$

The first expansion (3.32) can be proved similarly.  $\square$

**Proof of Property 6.** Let  $\psi_i$  be  $q$ -Grassmann variables commuting with  $M$ ,  $\psi_i^M = \sum_{k=1}^n \psi_k M_{ki}$  and  $I_1 = (i_1, \dots, i_m)$ ,  $I_2 = (i_{m+1}, \dots, i_n)$  such that  $1 \leq i_l \leq n$ . Using (3.18) we obtain

$$\begin{aligned} \psi_{i_1}^M \cdots \psi_{i_n}^M &= (\psi_{i_1}^M \cdots \psi_{i_m}^M) (\psi_{i_{m+1}}^M \cdots \psi_{i_n}^M) = \\ &= \sum_{\substack{K_1=(k_1 < \dots < k_m) \\ K_2=(k_{m+1} < \dots < k_n)}} \psi_{k_1} \cdots \psi_{k_m} \psi_{k_{m+1}} \cdots \psi_{k_n} \det_q(M_{K_1 I_1}) \det_q(M_{K_2 I_2}). \quad (3.40) \end{aligned}$$

where  $1 \leq k_1 < \dots < k_m \leq n$ ,  $1 \leq k_{m+1} < \dots < k_n \leq n$ . Note that  $\psi_{k_1} \cdots \psi_{k_m} \psi_{k_{m+1}} \cdots \psi_{k_n}$  does not vanish only if  $K_2 = \setminus K_1$ . In this case  $(k_1, \dots, k_m, k_{m+1}, \dots, k_n)$  is a permutation of  $(1, \dots, m, m+1, \dots, n)$  of length  $\sum_{l=1}^m (k_l - l)$ :

$$\psi_{i_1}^M \cdots \psi_{i_n}^M = \sum_{K=(k_1 < \dots < k_m)} (-q)^{-\sum_{l=1}^m (k_l - l)} \psi_1 \cdots \psi_n \det_q(M_{K I_1}) \det_q(M_{\setminus K I_2}). \quad (3.41)$$

Substituting  $i_l = l$  for  $l = 1, \dots, n$  we arrive at (3.33). If the matrix  $M$  is  $q$ -Manin the variables  $\psi_i^M$  are  $q$ -Grassmann. Substituting the formula (3.13) in the left hand side of (3.41) we obtain (3.34). Formula (3.35) can be proved in the same way.  $\square$

**Proof of Property 7.** We need to provide a counterexample, which is readily found in the  $2 \times 2$  case. Indeed, from formula (3.2) and the commutation relations of Remark (2.1)

we have, for the  $q$ -Manin matrix  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,

$$a(\det_q M) = a(ad - q^{-1}cb) = ada - qabc \neq \text{(in general)} ada - q^{-1}cba = (\det_q M)a.$$

□

**Remark 3.2.** It is easy to see that if  $M$  is a  $q$ -Manin matrix, then  $\widetilde{M}_{ij} = M_{n-i+1, n-j+1}$  is a  $q^{-1}$ -Manin matrix and one can also prove, that  $\det_q M = \det_{q^{-1}} \widetilde{M}$ . For example

$$\det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - q^{-1}cb = da - qbc = \det_{q^{-1}} \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \quad (3.42)$$

**Proof.** The fact that if  $M$  is a  $q$ -Manin matrix then  $\widetilde{M}$  is a  $q^{-1}$ -Manin matrix easy follows from the formulae (2.1), (2.2) (see also Remark 2.2). Let us apply Proposition 3.2 to the matrix  $\widetilde{M}$  and  $q^{-1}$ -Grassmann variables  $\widetilde{\psi}_i = \psi_{n+1-i}$ . The formula (3.18) gives us  $\widetilde{\psi}_1^M \cdots \widetilde{\psi}_n^M = \widetilde{\psi}_1 \cdots \widetilde{\psi}_n \det_{q^{-1}}(\widetilde{M})$ , where  $\widetilde{\psi}_j^M = \widetilde{\psi}_j \widetilde{M}_{ij}$ . Note that  $\widetilde{\psi}_j^M = \sum_i \psi_i M_{i, n+1-j} = \psi_{n+1-j}^M$ , where  $\psi_j^M = \sum_i \psi_i M_{ij}$  satisfy (3.18) with  $\psi_i$  and  $\det_q(M)$ . This yields

$$\begin{aligned} \widetilde{\psi}_1^M \cdots \widetilde{\psi}_n^M &= \psi_n^M \cdots \psi_1^M = (-q)^{-\frac{n(n-1)}{2}} \psi_1^M \cdots \psi_n^M = \\ &= (-q)^{-\frac{n(n-1)}{2}} \det_q(M) \psi_1 \cdots \psi_n = \det_q(M) \psi_n \cdots \psi_1 = \det_q(M) \widetilde{\psi}_1 \cdots \widetilde{\psi}_n. \end{aligned} \quad (3.43)$$

So we obtain  $\det_{q^{-1}}(\widetilde{M}) = \det_q(M)$ . □

### 3.3 The $q$ -Characteristic polynomial

Let us discuss now some subtleties in the definition of the characteristic polynomial for  $q$ -Manin matrices. In the classical case the characteristic polynomial of  $M$  was defined as  $\det(\lambda - M)$ , and the same definition (provided the column-determinant is used) hold for the  $q = 1$  case of the "ordinary" Manin matrices discussed in [5]. However, the fact that, for  $\lambda \in \mathbb{C}$  the matrix  $(\lambda - M)$  is *not* a  $q$ -Manin matrix clearly signals that the naïve generalization – (i.e., the  $q$ -determinant of  $(\lambda - M)$ ) – cannot be the correct one. To get a "good" definition of  $q$ -characteristic polynomial, we must rely on another property-definition of the characteristic polynomial of a matrix.

**Definition 4.** The  $q$ -characteristic polynomial for a matrix  $M$  is defined as follows:

$$\text{char}_q(\lambda, M) = \sum_{k=0}^n (-1)^k \lambda^{n-k} \sum_{I=(i_1 < i_2 < \dots < i_k)} \det_q(M_{II}) = \quad (3.44)$$

$$= \lambda^n - \lambda^{n-1} \text{tr } M + \dots + (-1)^n \det_q M, \quad (3.45)$$

that is, it is the weighted sum of principal  $q$ -minors. Here  $(M_{II})_{ab} = M_{i_a i_b}$  are principal submatrices of  $M$  of the size equal to the cardinality of  $I$ .



Clearly enough, for  $q = 1$  one gets the usual definition:  $\text{char}_{q=1}(\lambda, M) = \det_{q=1}(\lambda - M)$ .

An equivalent definition could be given as follows. Let  $\Lambda^k[\psi_1, \dots, \psi_n]$  be the subspace of the  $q$ -Grassmann algebra  $\mathbb{C}[\psi_1, \dots, \psi_n]$  consisting of elements of degree  $k$ . For a  $n \times n$  matrix  $M$  over  $\mathfrak{R}$  we define the (left) action of  $M$  on  $\mathfrak{R} \otimes \mathbb{C}[\psi_1, \dots, \psi_n]$  as  $M(r_0 \psi_{i_1} \psi_{i_2} \cdots \psi_{i_k}) = M(\psi_{i_1})M(\psi_{i_2}) \cdots M(\psi_{i_k})r_0$ , where  $r_0 \in \mathfrak{R}$  and  $M(\psi_j) = \sum_i \psi_i M_{ij}$ . The subspace  $\mathfrak{R} \otimes \Lambda^k[\psi_1, \dots, \psi_n]$  is invariant under this action and then the  $q$ -characteristic polynomial can be written in the following form

$$\text{char}_q(\lambda, M) = \sum_{k=0}^n \lambda^{n-k} (-1)^k \text{tr}_{\Lambda^k[\psi_1, \dots, \psi_n]} M. \quad (3.46)$$

We will show in this paper that the coefficients of the  $q$ -characteristic polynomial satisfy many important properties. Indeed, they enter  $q$ -analogues of the Newton, Cayley-Hamilton, and MacMahon-Wronski identities and have important applications in quantum integrable systems.

### 3.4 A $q$ -generalization of the Cramer formula and quasi-determinants

Here we consider a square  $n \times n$   $q$ -Manin matrix and present some relation between its  $q$ -determinant and the inverse matrix. In the commutative case they are reduced to the Cramer's formula. We formulate these relations in terms of Gelfand-Retakh-Wilson's quasi-determinants and apply the theory of quasi-determinants to the Gauss decomposition.

#### 3.4.1 Left adjoint matrix

First we introduce the left adjoint matrix in terms of  $(n-1) \times (n-1)$   $q$ -minors. The main aim of this subsection is to express the (left) inverse matrix through the adjoint matrix, or equivalently, through these  $q$ -minors.

**Proposition 3.4.** *Let  $M$  be a  $q$ -Manin matrix and  $M^{adj}$  be the matrix with the entries*

$$M_{sr}^{adj} = (-q)^{r-s} \det_q M_{\setminus r \setminus s}, \quad (3.47)$$

where  $M_{\setminus r \setminus s}$  is the  $(n-1) \times (n-1)$  submatrix of  $M$  defined by deleting  $r$ -th row and  $s$ -th column. Then

$$M^{adj} M = \det_q M \cdot 1_{n \times n}, \quad (3.48)$$

i.e.  $M^{adj}$  is a left adjoint matrix for the matrix  $M$ .

**Proof.** Using the formula (3.32) for the column expansion and vanishing of  $q$ -determinant for coincident columns (formula (3.27)) one gets

$$\sum_{r=1}^n M_{sr}^{adj} M_{rk} = \sum_{r=1}^n (-q)^{r-s} \det_q(M_{\setminus r \setminus s}) M_{rk} = \det_q \widetilde{M} = \delta_{sk} \det_q M, \quad (3.49)$$

where  $\widetilde{M}$  is the matrix obtained from  $M$  by the replacement the  $s$ -th column with the  $k$ -th one.  $\square$

**Example 3.4.** In the case  $n = 2$  the formula (3.48) is deduced as follows (in the notations (2.9))

$$\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} da - qbc & db - qbd \\ -q^{-1}ca + ac & -q^{-1}cb + ad \end{pmatrix} = \begin{pmatrix} \det_q M & 0 \\ 0 & \det_q M \end{pmatrix}.$$

**Corollary 3.4.1.** *Let  $M$  be a  $q$ -Manin matrix and suppose that its  $q$ -determinant  $\det_q M$  is invertible from the left, so that there exists an element (a left inverse of the  $q$ -determinant)  $(\det_q M)^{-1}$  such that  $(\det_q M)^{-1} \det_q M = 1$ . Then the product  $M^{-1} = (\det_q M)^{-1} M^{adj}$  is a left inverse of  $M$ , that is  $M^{-1}M = 1$ . The matrix  $M^{-1}$  consists of the following entries*

$$M_{sr}^{-1} = (\det_q M)^{-1} M_{sr}^{adj} = (-q)^{r-s} (\det_q M)^{-1} \det_q M_{\setminus r \setminus s}. \quad (3.50)$$

*In particular, the existence of a left inverse of the  $q$ -determinant  $\det_q(M)$  of a  $q$ -Manin matrix  $M$  implies the existence of a left inverse of  $M$ .*

Let us remark that the left invertibility of a  $q$ -Manin matrix  $M$  does not imply the left invertibility of  $\det_q M$ . On the other hand, neither left nor right invertibility of  $\det_q M$  nor both of them implies the right invertibility of  $M$ . Let us also remind that the left (right) invertibility of an element of some non-commutative algebra does not guarantee that the left (right) inverse is unique. In Corollary 3.4.1 we claim that there exists at least one inverse  $M^{-1} = (\det_q M)^{-1} M^{adj}$ .

### 3.4.2 Relation with the quasi-determinants

We will herewith recall a few constructions from the theory of quasi-determinants of I. Gelfand and V. Retakh and discuss their counterparts in the case of Manin matrices. It is fair to say that the general theoretical set-up of quasi-determinants developed by Gelfand, Retakh and collaborators (see, e.g., [20], [17], [19]) can be briefly presented as follows: *the basic facts of linear algebra can be reformulated with the only use of an inverse matrix*. Thus it can be extended to the non-commutative set-up and can be applied, for example, to some questions considered here. We must stress the difference between our set-up and the much more general one of [20]: we herewith consider *a special class* of matrices with non-commutative entries (the  $q$ -Manin matrices), and for this class we can extend theorems of linear algebra basically in the same form as in the commutative case, (in particular, as we have seen, there exists a well-defined notion of determinant). On the other hand, in [20] *generic matrices* are considered; thus there is no natural notion of the determinant, and facts of linear algebra are not exactly presents in the same form as in the commutative case.

Let us recall ([20] definition 1.2.2) that the  $(p, q)$ -th quasi-determinant  $|A|_{pq}$  of an invertible matrix  $A$  is defined as  $|A|_{pq} = (A_{qp}^{-1})^{-1}$ , i.e. as the inverse to the  $(q, p)$ -element

of the matrix inverse to  $A$ . It is also denoted by:

$$|A|_{pq} = \begin{vmatrix} A_{11} & A_{12} & \dots & \dots & \dots & A_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \boxed{A_{pq}} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}, \quad (3.51)$$

Let  $M$  be a  $n \times n$   $q$ -Manin matrix. Let  $M$  and  $\det_q M$  be two-sided invertible. Then their inverse are unique and the entries of  $M^{-1}$  are expressed by the formula (3.50). Moreover, in the case when  $\det_q M$  is two-sided invertible the left (right) invertibility of the entry  $M_{sr}^{-1}$  is equivalent to the left (right) invertibility of the determinant  $\det_q(M_{\setminus r \setminus s})$ . Thus the following lemma holds.

**Lemma 3.5.** *If the matrix  $M$  and the elements  $\det_q M$  and an entry  $M_{sr}^{-1}$  are two-sided invertible (for some  $1 \leq s, r \leq n$ ) then the  $(r, s)$ -th quasi-determinant  $|M|_{rs} = (M_{sr}^{-1})^{-1}$  is expressed by the formula*

$$|M|_{rs} = (-q)^{s-r} (\det_q(M_{\setminus r \setminus s}))^{-1} \det_q(M). \quad (3.52)$$

Using the formula (3.52) we can generalize the notion of quasi-determinant to the case when  $\det_q M$  is not two-sided invertible.

**Definition 5.** *Let the determinant  $\det_q(M_{\setminus r \setminus s})$  be two-sided invertible. Then the element  $|M|_{rs}$  defined by the formula (3.52) is called the  $(r, s)$ -th quasi-determinant  $|M|_{rs}$  for a  $q$ -Manin matrix  $M$ .*

The following lemma is often useful in applications of quasi-determinants to determinants. It holds thanks to the Cramer rule for  $q$ -Manin matrices.

**Proposition 3.6.** *(c.f. [17, 18]). Assume  $M$  is a  $q$ -Manin matrix, then*

$$\begin{aligned} & \det_q \begin{pmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{pmatrix} = \\ &= M_{nn} \begin{vmatrix} \boxed{M_{n-1 \ n-1}} & M_{n-1 \ n} \\ M_{nn-1} & M_{nn} \end{vmatrix} \dots \begin{vmatrix} \boxed{M_{22}} & \dots & M_{2n} \\ \vdots & \ddots & \vdots \\ M_{n2} & \dots & M_{nn} \end{vmatrix} \begin{vmatrix} \boxed{M_{11}} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & M_{nn} \end{vmatrix} = \\ &= M_{11} \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & \boxed{M_{22}} \end{vmatrix} \dots \begin{vmatrix} M_{11} & \dots & M_{1n-1} \\ \vdots & \ddots & \vdots \\ M_{n-11} & \dots & \boxed{M_{n-1 \ n-1}} \end{vmatrix} \begin{vmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & M_{22} & \dots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \dots & \boxed{M_{nn}} \end{vmatrix}, \end{aligned} \quad (3.53)$$

if the corresponding quasi-determinants are defined.

**Proof.** By the formula (3.52) we obtain

$$\left| \begin{array}{ccc} \boxed{M_{kk}} & \dots & M_{kn} \\ \vdots & \ddots & \vdots \\ M_{nk} & \dots & M_{nn} \end{array} \right| = \left[ \det_q \begin{pmatrix} M_{k+1,k+1} & \dots & M_{k+1,n} \\ \vdots & \ddots & \vdots \\ M_{n,k+1} & \dots & M_{nn} \end{pmatrix} \right]^{-1} \det_q \begin{pmatrix} M_{kk} & \dots & M_{kn} \\ \vdots & \ddots & \vdots \\ M_{nk} & \dots & M_{nn} \end{pmatrix}.$$

Substituting it to (3.53) we obtain the identity for the first formula. The second formula is proved in the same way.  $\square$

### 3.4.3 Gauss decomposition and $q$ -determinant

Here we show that the  $q$ -determinant of a  $q$ -Manin matrix can be expressed via the diagonal part of the Gauss decomposition exactly in the same way as in the commutative case.

**Proposition 3.7.** *Let  $M$  be a  $q$ -Manin matrix. If it is factorized into Gauss form*

$$M = \begin{pmatrix} 1 & & x_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ z_{\beta\alpha} & & 1 \end{pmatrix} \quad (3.54)$$

where  $y_1, \dots, y_n$  are two-sided invertible then

$$\det_q M = y_n \dots y_1. \quad (3.55)$$

Analogously if

$$M = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ x'_{\alpha\beta} & & 1 \end{pmatrix} \begin{pmatrix} y'_1 & & 0 \\ & \ddots & \\ 0 & & y'_n \end{pmatrix} \begin{pmatrix} 1 & & z'_{\beta\alpha} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad (3.56)$$

with two-sided invertible  $y'_1, \dots, y'_n$  then

$$\det_q M = y'_1 \dots y'_n. \quad (3.57)$$

**Example 3.5.** In the case  $n = 2$  we have

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}, \quad (3.58)$$

$$\det_q(M) = ad - q^{-1}cb = da - qbc = d(a - bd^{-1}c), \quad (3.59)$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}, \quad (3.60)$$

$$\det_q(M) = ad - q^{-1}cb = a(d - ca^{-1}b). \quad (3.61)$$

Remark that in the second of these example one only needs the first column  $q$ -commutativity relation.

**Proof.** First we note that the decomposition (3.54) for the following submatrices holds with the same  $y_i$ :

$$M_{(k)} = \begin{pmatrix} M_{kk} & \dots & M_{kn} \\ \vdots & \ddots & \vdots \\ M_{nk} & \dots & M_{nn} \end{pmatrix} = \begin{pmatrix} 1 & & x_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_k & & 0 \\ & \ddots & \\ 0 & & y_n \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ z_{\beta\alpha} & & 1 \end{pmatrix}.$$

Then note that a left (right) triangular matrix with non-commutative entries and with unity on the diagonal is two-sided invertible and its inverse is a left (right) triangular matrix with non-commutative entries and with unity on the diagonal. Hence the inverse of  $M_{(k)}$  has the form

$$M_{(k)}^{-1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \tilde{z}_{\beta\alpha} & & 1 \end{pmatrix} \begin{pmatrix} y_k^{-1} & & 0 \\ & \ddots & \\ 0 & & y_n^{-1} \end{pmatrix} \begin{pmatrix} 1 & & \tilde{x}_{\alpha\beta} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \quad (3.62)$$

Taking into account  $(M_{(k)}^{-1})_{11} = y_k^{-1}$  we obtain the corresponding quasi-determinant

$$\begin{vmatrix} \boxed{M_{kk}} & \dots & M_{kn} \\ \vdots & \ddots & \vdots \\ M_{nk} & \dots & M_{nn} \end{vmatrix} = y_k. \quad (3.63)$$

Thus Proposition 3.6 and  $M_{nn} = y_n$  imply (3.55). The formula (3.57) is proved similarly.  $\square$

**Remark 3.3.** The Gauss components  $x_{\alpha\beta}$ ,  $y_k$ ,  $z_{\beta\alpha}$ ,  $\tilde{x}_{\alpha\beta}$ ,  $\tilde{y}_k$ ,  $\tilde{z}_{\beta\alpha}$  are unique and can be expressed in terms of quasideterminants such as (3.63) (see [18, 20] for details). Thus they can be expressed in terms of  $q$ -minors of  $M$ .

## 4 $q$ -Minors of a $q$ -Manin matrix and its inverse

In this section we discuss several formulae holding (in a manner substantially similar to that of the classical case) for  $q$ -Manin matrices. In particular, in the first Subsection we prove the  $q$ -analogue of the Jacobi ratio theorem (which expresses minors of inverse matrix in terms of the minors of the original matrix). Building on this, in the second Subsection, we recover ( $q$ -analogues of several statements of linear algebra, such as the so-called Dodgson (or Lagrange – Desnanot– Jacobi– Lewis Carroll) condensation formula, the Schur decomposition formula, the Sylvester identities, and (although limiting ourselves to the simplest non-trivial case), the Plücker relations. We also show that the inverse of a  $q$ -Manin matrix is a  $q^{-1}$ -Manin matrix.

## 4.1 Jacobi ratio theorem

This subsection is devoted to the proof of the  $q$  analogue of the Jacobi ratio theorem. We deem useful to preliminarily prove a few propositions and lemmas, to be used in the main proof. They also might be of independent interest. In particular, we obtain a formula for the right inverse of a two-sided invertible  $q$ -Manin matrix.

### 4.1.1 Preliminary propositions. Right inverse for the $q$ -determinant of a $q$ -Manin matrix

**Lemma 4.1.** *Assume that an element  $\psi$  commute with all the entries of a two-sided invertible matrix  $M$ ; then  $\psi$  commute with all the entries of its inverse  $M^{-1}$ .*

**Proof.** Multiplying the relation  $\psi M_{ij} = M_{ij} \psi$  by  $M_{jk}^{-1}$  on the right,  $M_{li}^{-1}$  on the left and taking summation over  $j$  and  $i$  we obtain  $M_{lk}^{-1} \psi = \psi M_{lk}^{-1}$ .  $\square$

**Lemma 4.2.** *Let  $\psi_1, \dots, \psi_n$  be  $q$ -Grassmann variables (or at least non-commuting elements satisfying  $\psi_i \psi_j = -q \psi_j \psi_i$  for  $i < j$ ). Then*

$$\psi_{k_{\tau(m)}} \cdots \psi_{k_{\tau(1)}} = (-q)^{\text{inv}(\tau)} \psi_{k_m} \cdots \psi_{k_1}, \quad (4.1)$$

where  $k_1 < \dots < k_m$  and  $\tau \in \mathfrak{S}_m$ .

**Proof.** Let  $\sigma$  be the longest element of  $\mathfrak{S}_m$ , i.e.  $\sigma(i) = m + 1 - i$ . Then the use of the formula (3.12) and  $\text{inv}(\tau\sigma) = \text{inv}(\sigma) - \text{inv}(\tau)$  yields (4.1).

**Proposition 4.3.** *Consider a two-sided invertible  $q$ -Manin matrix  $M$ . Let  $\psi_i$  be  $q$ -Grassmann variables commuting with entries of  $M$  and let  $\psi_j^M = \sum_i \psi_i M_{ij}$ . Then for a general multi-index  $J = (j_1, j_2, \dots, j_m)$  we have*

$$\psi_{j_m} \cdots \psi_{j_1} = \sum_{L=(l_1 < \dots < l_m)} \psi_{l_m}^M \cdots \psi_{l_1}^M \det_{q^{-1}}(M_{LJ}^{-1}), \quad (4.2)$$

where  $M_{LJ}^{-1} = (M^{-1})_{LJ}$  is the corresponding  $m \times m$  submatrix of  $M^{-1}$ .

**Proof.** Using  $\psi_j = \sum_l \psi_l^M M_{lj}^{-1}$  and taking into account that  $\psi_j$  commute with the entries of  $M^{-1}$  (Lemma 4.1) we obtain, at first

$$\psi_{j_m} \cdots \psi_{j_1} = \left( \sum_{l_m} \psi_{l_m}^M M_{l_m j_m}^{-1} \right) \psi_{j_{m-1}} \cdots \psi_{j_1} \stackrel{\text{Lemma: } [M_{ij}^{-1}, \psi_k]=0}{=} \sum_{l_m} \psi_{l_m}^M \psi_{j_{m-1}} \cdots \psi_{j_1} M_{l_m j_m}^{-1}. \quad (4.3)$$

Then, the second step and further iterations yield

$$\begin{aligned} \sum_{l_m} \psi_{l_m}^M \left( \sum_{l_{m-1}} \psi_{l_{m-1}}^M M_{l_{m-1} j_{m-1}}^{-1} \right) \psi_{l_{m-2}} \cdots \psi_{j_1} M_{l_m j_m}^{-1} &\stackrel{\text{Lemma: } [M_{ij}^{-1}, \psi_k]=0}{=} \\ &= \sum_{l_m} \sum_{l_{m-1}} \psi_{l_1}^M \psi_{l_2}^M \psi_{j_{m-3}} \cdots \psi_{j_1} M_{l_{m-1} j_{m-1}}^{-1} M_{l_m j_m}^{-1}. \end{aligned} \quad (4.4)$$

We obtain making this transformation  $m$  times :

$$\sum_{l_1, \dots, l_m} \psi_{l_m}^M \cdots \psi_{l_1}^M M_{l_1 j_1}^{-1} \cdots M_{l_m j_m}^{-1}. \quad (4.5)$$

But using the Manin's property (Proposition 2.1 page 8 ) one has that the elements  $\psi_{l_1}^M, \dots, \psi_{l_m}^M$  are  $q$ -Grassmann variables, so we obtain finally

$$\sum_{l_1 < l_2 < \dots < l_m} \sum_{\tau \in \mathfrak{S}_m} \psi_{l_{\tau(m)}}^M \cdots \psi_{l_{\tau(1)}}^M M_{l_{\tau(1)} j_1}^{-1} \cdots M_{l_{\tau(m)} j_m}^{-1}. \quad (4.6)$$

Using Lemma 4.2 we can rewrite it as

$$\begin{aligned} \sum_{\tau \in \mathfrak{S}_m} \sum_{l_1 < l_2 < \dots < l_m} (-q^{-1})^{-\text{inv}(\tau)} \psi_{l_m}^M \cdots \psi_{l_1}^M M_{\tau(l_1) j_1}^{-1} \cdots M_{\tau(l_m) j_m}^{-1} = \\ = \sum_{L=(l_1 < \dots < l_m)} \psi_{l_m}^M \cdots \psi_{l_1}^M \det_{q^{-1}}(M_{LJ}^{-1}). \end{aligned}$$

Thus the formula (4.2) is proved.  $\square$

**Remark 4.1.** Let us note that two-sided invertibility of  $M$  is crucial since we need it to use Lemma 4.1.

**Proposition 4.4.** *Let  $M$  be a two-sided invertible  $q$ -Manin matrix and consider a multi-index  $K = (k_1 < k_2 < \dots < k_m)$ . Then for a general multiindex of a cardinality  $m$ , say,  $J = (j_1, j_2, \dots, j_m)$  we have*

$$\sum_{L=(l_1 < l_2 < \dots < l_m)} \det_q(M_{KL}) \det_{q^{-1}}(M_{LJ}^{-1}) = \sum_{\tau \in \mathfrak{S}_m} (-q)^{\text{inv}(\tau)} \delta_{k_{\tau(1)}}^{j_1} \cdots \delta_{k_{\tau(m)}}^{j_m}, \quad (4.7)$$

where  $M_{LJ}^{-1} = (M^{-1})_{LJ}$  is a submatrix of  $M^{-1}$ .

**Proof.** Multiplying the relation (3.19) by  $(-q)^{-m(m-1)/2}$  and re-ordering  $q$ -Grassmann variables we obtain

$$\psi_{l_m}^M \cdots \psi_{l_1}^M = \sum_{K=(k_1 < \dots < k_m)} \psi_{k_m} \cdots \psi_{k_1} \det_q(M_{KL}). \quad (4.8)$$

Substituting the formula (4.8) in (4.2) we obtain

$$\psi_{j_m} \cdots \psi_{j_1} = \sum_{\substack{L=(l_1 < l_2 < \dots < l_m) \\ K=(k_1 < \dots < k_m)}} \psi_{k_m} \cdots \psi_{k_1} \det_q(M_{KL}) \det_{q^{-1}}(M_{LJ}^{-1}). \quad (4.9)$$

Comparing the coefficients at  $\psi_{k_m} \cdots \psi_{k_1}$  with  $k_1 < \dots < k_m$  in sides of (4.9) we see that the expression in the left hand side of (4.7) does not vanish only if  $K$  is a permutation of



$J$ , that is if  $(j_1, \dots, j_m) = (k_{\tau(1)}, \dots, k_{\tau(m)})$  for some  $\tau \in \mathfrak{S}_m$ . The use of Lemma 4.2 yields in this case

$$(-q)^{\text{inv}(\tau)} \psi_{k_m} \cdots \psi_{k_1} = \sum_{\substack{L=(l_1 < l_2 < \dots < l_m) \\ K=(k_1 < \dots < k_m)}} \psi_{k_m} \cdots \psi_{k_1} \det_q(M_{KL}) \det_{q^{-1}}(M_{LJ}^{-1}). \quad (4.10)$$

Hence the expression in the left hand side of (4.7) in this case is equal to  $(-q)^{\text{inv}(\tau)}$ . This gives us the formula (4.7).  $\square$

In the  $q = 1$  commutative case Proposition 4.4 is a corollary of the Cauchy-Binet formula (3.30) applied to the identity  $MM^{-1} = 1$ . In spite of the fact that the entries of  $M$  and  $M^{-1}$  do not commute in general the formula (3.30) works also for  $q$ -Manin matrices with  $\det_{q^{-1}}(M^{-1})$  and  $\det_q(M)$ .

Considering the formula (4.7) in the case  $K = J = (1, 2, \dots, n)$  we obtain the following corollary.

**Corollary 4.4.1.** *If  $M$  is two-sided invertible  $q$ -Manin matrix then the  $q^{-1}$ -determinant of its inverse  $M^{-1}$  is right inverse of  $\det_q(M)$ :*

$$\det_q(M) \det_{q^{-1}}(M^{-1}) = 1. \quad (4.11)$$

*In particular, if the  $q$ -determinant of a two-sided invertible  $q$ -Manin matrix is right invertible.*

**Remark 4.2.** Let  $M$  be an  $n \times n$   $q$ -Manin matrix and  $\widetilde{M}$  be a  $n \times n$  matrix satisfying  $\sum_{j=1}^n \widetilde{M}_{jk} M_{ij} = \sum_{j=1}^n M_{jk} \widetilde{M}_{ij} = \delta_{ik}$ . Then one can derive the following formulae in similar way:

$$\sum_{L=(l_1 < l_2 < \dots < l_m)} \det_{q^{-1}}(\widetilde{M}_{LJ}) \det_q(M_{KL}) = \sum_{\tau \in \mathfrak{S}_m} (-q)^{\text{inv}(\tau)} \delta_{k_{\tau(1)}}^{j_1} \cdots \delta_{k_{\tau(m)}}^{j_m}, \quad (4.12)$$

$$\det_{q^{-1}} \widetilde{M} \det_q M = 1. \quad (4.13)$$

**Remark 4.3.** Is it also true that  $\det_{q^{-1}}(M^{-1}) \det_q M = 1$  for a two-sided invertible  $q$ -Manin matrix  $M$ ? In other words, does the two-sided invertibility of a  $q$ -Manin matrix implies the left invertibility of its  $q$ -determinant? We don't know an answer to this question.

#### 4.1.2 Jacobi ratio theorem for $q$ -Manin matrices

Here we will formulate and prove a  $q$ -analogue of the Jacobi ratio theorem.

Let  $J = (j_1, \dots, j_m)$  where  $1 \leq j_l \leq n$ . We define the symbol  $\epsilon_J^{q^{-1}} = \epsilon_{j_1, \dots, j_m}^{q^{-1}}$  by the formulae  $\epsilon_{\dots, j, \dots, j, \dots}^{q^{-1}} = 0$  and

$$\epsilon_{k_{\tau(1)}, \dots, k_{\tau(m)}}^{q^{-1}} = (-q)^{\text{inv}(\tau)} \quad \text{for } 1 \leq k_1 < \dots < k_m \leq n \text{ and } \tau \in \mathfrak{S}_m. \quad (4.14)$$

It generalizes the symbol defined in (3.14) for  $q^{-1}$  and  $m \leq n$ .

**Theorem 4.5.** *Let  $M$  be a two-sided invertible  $q$ -Manin matrix. Consider multi-indices  $I = (i_1 < i_2 < \dots < i_m)$  and  $J = (j_1, \dots, j_m)$ . Then the minors of  $M^{-1}$  can be expressed through the complementary minors of  $M$  by the formula*

$$\det_q(M) \det_{q^{-1}}(M_{IJ}^{-1}) = (-q)^{\sum_{i=1}^m (j_i - i_i)} \varepsilon_J^{q^{-1}} \det_q(M_{\setminus J \setminus I}), \quad (4.15)$$

where  $M_{LJ}^{-1} = (M^{-1})_{LJ}$  and, as usual, we denote by  $\setminus I$  the multi-index obtained by deleting  $i_1, \dots, i_m$  from the sequence  $(1, 2, 3, \dots, n)$ .<sup>5</sup> In particular

$$\det_q(M) \det_{q^{-1}}(M_{IJ}^{-1}) = (-q)^{\sum_{i=1}^m (j_i - i_i)} \det_q(M_{\setminus J \setminus I}), \quad \text{if } j_1 < j_2 < \dots < j_m, \quad (4.16)$$

$$\det_q(M) \det_{q^{-1}}(M_{IJ}^{-1}) = 0 \quad \text{if } j_a = j_b \text{ for some } a \neq b. \quad (4.17)$$

Let us remark that in the case  $I = J = (1, 2, \dots, n)$  the Jacobi ratio formula (4.16) coincides with (4.11).

**Proof.** Let us rewrite the  $q$ -Laplace identity (3.34) in the form

$$\varepsilon_{\setminus I \oplus L}^q \det_q(M) = \sum_{K=(k_1 < k_2 < \dots < k_m)} (-q)^{-\sum_{l=1}^{n-m} (k_{m+l} - l)} \det_q(M_{\setminus K \setminus I}) \det_q(M_{KL}), \quad (4.18)$$

where  $L = (l_1 < l_2 < \dots < l_m)$  is an arbitrary multi-index,  $\varepsilon^q$  is the usual  $q$ -epsilon-symbol defined by (3.14); multi-index  $(\setminus I \oplus L) = (i_{m+1}, i_{m+2}, \dots, i_n, l_1, l_2, \dots, l_m)$  where  $i_{m+1}, i_{m+2}, \dots, i_n$  are integers such that  $\setminus I = (i_{m+1}, i_{m+2}, \dots, i_n)$ . Notice that

$$\sum_{l=1}^{n-m} (k_{m+l} - l) = \sum_{l=m+1}^n k_l - \sum_{l=1}^{n-m} l = \sum_{l=1}^n l - \sum_{l=1}^m k_l - \sum_{l=1}^{n-m} l = \sum_{l=n-m+1}^n l - \sum_{l=1}^m k_l. \quad (4.19)$$

Let us multiply the identity (4.18) by  $\det_{q^{-1}}(M_{LJ}^{-1})$  from the right, take a summation over  $L = (l_1 < l_2 < \dots < l_m)$  and transform the right hand side using (4.19) and Proposition 4.4:

$$\begin{aligned} & \sum_{L=(l_1 < l_2 < \dots < l_m)} \varepsilon_{\setminus I \oplus L}^q \det_q(M) \det_{q^{-1}}(M_{LJ}^{-1}) = \\ &= \sum_{\substack{L=(l_1 < l_2 < \dots < l_m) \\ K=(k_1 < k_2 < \dots < k_m)}} (-q)^{\sum_{l=1}^m k_l - \sum_{l=n-m+1}^n l} \det_q(M_{\setminus K \setminus I}) \det_q(M_{KL}) \det_{q^{-1}}(M_{LJ}^{-1}) = \\ &= \sum_{\substack{K=(k_1 < k_2 < \dots < k_m) \\ \tau \in \mathfrak{S}_m}} (-q)^{\sum_{l=1}^m k_l - \sum_{l=n-m+1}^n l} \det_q(M_{\setminus K \setminus I}) (-q)^{\text{inv}(\tau)} \delta_{k_{\tau(1)}}^{j_1} \dots \delta_{k_{\tau(m)}}^{j_m}. \end{aligned} \quad (4.20)$$

---

<sup>5</sup>If  $J = (\dots, j, \dots, j, \dots)$  the expression  $\det_q(M_{\setminus J \setminus I})$  is not defined, but  $\varepsilon_J^{q^{-1}} = 0$  in this case and then we can formally consider  $\varepsilon_J^{q^{-1}} \det_q(M_{\setminus J \setminus I}) = 0$ . In the case  $I = J = \emptyset$  it is natural to define  $\det_q(M_{IJ}) = 1$  for an arbitrary matrix  $M$ .

The summation in the right hand side gives us  $(-q)^{\sum_{l=1}^m j_l - \sum_{l=n-m+1}^n l} \varepsilon_J^{q^{-1}} \det_q(M_{\setminus J \setminus I})$ . At the left hand side we have  $\varepsilon_{\setminus I \oplus L}^q = 0$ , unless  $L = I$ . So the formula (4.20) transforms to

$$\varepsilon_{\setminus I \oplus I}^q \det_q(M) \det_{q^{-1}}(M_{IJ}^{-1}) = (-q)^{\sum_{l=1}^m j_l - \sum_{l=n-m+1}^n l} \varepsilon_J^{q^{-1}} \det_q(M_{\setminus J \setminus I}), \quad (4.21)$$

Recalling that (see (3.16) page 14)

$$\varepsilon_{\setminus I \oplus I}^q = (-q)^{\sum_{l=1}^m i_l - \sum_{l=n-m+1}^n l}, \quad (4.22)$$

we come to the formula (4.15).  $\square$

**Remark 4.4.** The factor in the theorem can be rewritten via the complementary indices:

$$\sum_{l=1}^m (j_l - i_l) = - \sum_{l=m+1}^n (j_l - i_l) \quad (4.23)$$

since  $\sum_{l=1}^n i_l = \sum_{l=1}^n j_l = 1 + 2 + \dots + n$ .

## 4.2 Corollaries

Theorem 4.5 has a number of important consequences. In particular, the following one is the key to prove the fact that the inverses of some  $q$ -Manin matrices are  $q^{-1}$ -Manin matrices.

### 4.2.1 Lagrange-Desnanot-Jacobi-Lewis Carroll formula

Let us discuss the special case of the  $q$ -Jacobi ratio theorem 4.5 for indices of length two. It is interesting by many reasons. Besides the mentioned application (see the next subsubsection) it plays role in wide range of questions [12].

Recall that the Lagrange-Desnanot-Jacobi-Lewis Carroll formula reads:

$$\det(M_{\setminus j \setminus i}) \det(M_{\setminus l \setminus k}) - \det(M_{\setminus j \setminus k}) \det(M_{\setminus l \setminus i}) = \det M \det(M_{\setminus (jl) \setminus (ik)}), \quad (4.24)$$

where  $M$  is a matrix over  $\mathbb{C}$ ,  $j < l$  and  $i < k$ . According to ([2] page 111), Lagrange found this identity for  $n = 3$ , Desnanot proved it for  $n \leq 6$ , Jacobi proved his general theorem, C. L. Dodgson – better known as Lewis Carroll – used it to derive an algorithm for calculating determinants that required only  $2 \times 2$  determinants (“Dodgson’s condensation” method [11]).

Considering the case  $m = 2$  of Theorem 4.5 we obtain the following non-commutative  $q$ -generalization of this formula.

**Proposition 4.6.** *Consider a two-sided invertible  $q$ -Manin matrix  $M$  (i.e.  $\exists M^{-1} : M^{-1}M = MM^{-1} = 1$ ). Then for  $1 \leq i_1 < i_2 \leq n$ ,  $1 \leq j \leq n$  and  $1 \leq j_1 < j_2 \leq n$*

$$\det_q M (M_{i_1 j}^{-1} M_{i_2 j}^{-1} - q M_{i_2 j}^{-1} M_{i_1 j}^{-1}) = 0, \quad (4.25)$$

$$\det_q M (M_{i_1 j_1}^{-1} M_{i_2 j_2}^{-1} - q M_{i_2 j_1}^{-1} M_{i_1 j_2}^{-1}) = (-q)^{j_1 + j_2 - i_1 - i_2} \det_q (M_{\setminus (j_1 j_2) \setminus (i_1 i_2)}), \quad (4.26)$$

$$\det_q M (M_{i_1 j_2}^{-1} M_{i_2 j_1}^{-1} - q M_{i_2 j_2}^{-1} M_{i_1 j_1}^{-1}) = (-q)^{j_1 + j_2 - i_1 - i_2 + 1} \det_q (M_{\setminus (j_1 j_2) \setminus (i_1 i_2)}), \quad (4.27)$$

where  $M_{ij}^{-1}$  are entries of the inverse matrix  $M^{-1}$  and  $M_{\setminus(j_1 j_2) \setminus (i_1 i_2)}$  as usual is the  $(n-2) \times (n-2)$  matrix obtained from  $M$  deleting rows  $j_1, j_2$  and columns  $i_1, i_2$ .

**Remark 4.5.** In the commutative case, i.e.  $q = 1$  and  $[M_{ij}, M_{ij}] = 0$ , one has  $\det(M_{\setminus j \setminus i}) = (-1)^{i+j} \det M \cdot M_{ij}^{-1}$  and then both formulae (4.26) and (4.27) imply the formula (4.24) of Lagrange, Desnanot, Jacobi and Lewis Carroll.

**Remark 4.6.** Using the formula for the left adjoint matrix  $M^{adj} = \det_q M \cdot M^{-1}$  one can rewrite the relations (4.25)–(4.27) in the form

$$M_{i_1 j}^{adj} M_{i_2 j}^{-1} - q M_{i_2 j}^{adj} M_{i_1 j}^{-1} = 0, \quad (4.28)$$

$$M_{i_1 j_1}^{adj} M_{i_2 l_2}^{-1} - q M_{i_2 j_1}^{adj} M_{i_1 l_2}^{-1} = (-q)^{j_1 + l_2 - i_1 - i_2} \det_q (M_{\setminus(j_1 l_2) \setminus (i_1 i_2)}), \quad (4.29)$$

$$M_{i_1 l_2}^{adj} M_{i_2 j_1}^{-1} - q M_{i_2 l_2}^{adj} M_{i_1 j_1}^{-1} = (-q)^{j_1 + l_2 - i_1 - i_2 + 1} \det_q (M_{\setminus(j_1 l_2) \setminus (i_1 i_2)}), \quad (4.30)$$

where  $1 \leq i_1 < i_2 \leq n$ ,  $1 \leq j \leq n$  and  $1 \leq j_1 < j_2 \leq n$ . The formulae (4.28)–(4.30) are still valid if the matrix  $M$  is invertible *only* from the right, where  $M^{-1}$  is some right inverse:  $MM^{-1} = 1$ , (see Appendix B).

#### 4.2.2 The inverse of a $q$ -Manin matrix is a $q^{-1}$ -Manin matrix

In [4],[5] it was proved that the inverse of a two-sided invertible Manin matrix (the case  $q = 1$ ) is again a Manin matrix. There, it was also shown that this fact has a series of applications. Here we present its analogue for  $q$ -Manin matrices.

Consider a right invertible  $q$ -Manin matrix  $M$  with left invertible  $q$ -determinant  $\det_q M$ . The left invertibility of  $\det_q M$  implies the left invertibility of  $M$  (Corollary 3.4.1). Then according to the Corollary 4.4.1 the two-sided invertibility of  $M$  implies the right invertibility of  $\det_q M$ . In this way that the matrix  $M$  and its  $q$ -determinant  $\det_q M$  are two-sided invertible; i.e., there exist a matrix  $M^{-1}$  and an element  $(\det_q M)^{-1}$  such that

$$M^{-1}M = M^{-1}M = 1_{n \times n} \quad (4.31)$$

$$(\det_q M)^{-1} \det_q(M) = \det_q(M)(\det_q M)^{-1} = 1, \quad (4.32)$$

Notice also that due to the corollaries 3.4.1, 4.4.1 they are related as

$$M^{-1} = (\det_q M)^{-1} M^{adj}, \quad (4.33)$$

$$(\det_q M)^{-1} = \det_{q^{-1}}(M^{-1}). \quad (4.34)$$

**Theorem 4.7.** *If a  $q$ -Manin matrix  $M$  is invertible from the right and its  $q$ -determinant  $\det_q M$  is invertible from the left then the inverse matrix  $M^{-1}$  is a  $q^{-1}$ -Manin matrix.*

**Proof.** Due to the left invertibility of the  $q$ -determinant of  $M$  one can multiply the equations (4.25), (4.26), (4.27) by  $(\det_q M)^{-1}$  from the left. In the first equation we obtain

$$M_{i_1 j}^{-1} M_{i_2 j}^{-1} - q M_{i_2 j}^{-1} M_{i_1 j}^{-1} = 0, \quad (4.35)$$

where  $i_1 < i_2$ . Comparing the last two equations one gets

$$M_{i_1 j_1}^{-1} M_{i_2 j_2}^{-1} - q M_{i_2 j_1}^{-1} M_{i_1 j_2}^{-1} = -q^{-1} M_{i_1 j_2}^{-1} M_{i_2 j_1}^{-1} + M_{i_2 j_2}^{-1} M_{i_1 j_1}^{-1}. \quad (4.36)$$

where  $i_1 < i_2$ ,  $j_1 < j_2$ . So that we derive the commutation relations for the entries of a  $q^{-1}$ -Manin matrix (see the formulae (2.1), (2.2)).  $\square$

### 4.2.3 Schur complements

Here we apply Theorem 4.5 to the Schur complements. Let  $M$  be an arbitrary  $n \times n$  matrix over a non-commutative ring, which we consider partitioned in four blocks:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (4.37)$$

where  $A$ ,  $B$ ,  $C$  and  $D$  are  $m \times m$ ,  $m \times (n - m)$ ,  $(n - m) \times m$  and  $(n - m) \times (n - m)$  matrices respectively.

Suppose first that the matrix  $M$  is left-invertible and let  $M^{-1}$  be its left inverse, to be analogously partitioned into

$$M^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}. \quad (4.38)$$

Suppose also that the matrices  $A$  and  $D$  are right-invertible. Then  $\tilde{A}(A - BD^{-1}C) = 1$  and  $\tilde{D}(D - CA^{-1}B) = 1$ , where  $A^{-1}$  and  $D^{-1}$  are right inverses of  $A$  and  $D$  respectively.

If the matrix  $M$  is right-invertible with right inverse of the form (4.38) and the matrices  $A$  and  $D$  are left-invertible then  $(A - BD^{-1}C)\tilde{A} = 1$  and  $(D - CA^{-1}B)\tilde{D} = 1$ , where  $A^{-1}$  and  $D^{-1}$  are left inverse of  $A$  and  $D$  respectively. So finally we have the following lemma.

**Lemma 4.8.** *If  $M$ ,  $A$  and  $D$  are two-sided invertible then the matrices  $\tilde{A}$  and  $\tilde{D}$  defined from (4.38) are also two-sided invertible and*

$$\tilde{A}^{-1} = A - BD^{-1}C, \quad \tilde{D}^{-1} = D - CA^{-1}B. \quad (4.39)$$

**Definition 6.** The matrices  $A - BD^{-1}C$  and  $D - CA^{-1}B$  are called *Schur complements* of the blocks  $D$  and  $A$  respectively.

In terms of multi-indices we have

$$A = M_{II}, \quad D = M_{\setminus I \setminus I}, \quad \tilde{A} = M_{II}^{-1}, \quad \tilde{D} = M_{\setminus I \setminus I}^{-1}, \quad (4.40)$$

where  $I = (1, \dots, m)$ ,  $\setminus I = (m + 1, \dots, n)$ . Using (4.39) and (4.40) we can apply Theorem 4.5 (the case  $I = J = (1, \dots, m)$ ) to the Schur complements.

**Proposition 4.9.** *Let  $M$  be a  $q$ -Manin matrix and let  $A$  and  $D$  be its submatrices defined by the formula (4.37) (which are also  $q$ -Manin). Suppose  $M$ ,  $A$  and  $D$  are right-invertible and their  $q$ -determinants  $\det_q M$ ,  $\det_q A$  and  $\det_q D$  are left-invertible (hence they all are two-sided invertible). Then the Schur complements (4.39) are  $q$ -Manin matrices as well and their  $q$ -determinants satisfy the relations*

$$\det_q M = \det_q D \det_q (A - BD^{-1}C), \quad \det_q M = \det_q A \det_q (D - CA^{-1}B). \quad (4.41)$$

**Proof.** In order to prove that Schur complements are  $q$ -Manin we show that the matrices  $\tilde{A}$  and  $\tilde{D}$  defined by the formula (4.38) satisfy the condition of Theorem 4.7 (replacing  $q$  by  $q^{-1}$ ). First let us note that the matrix  $M$  satisfies the conditions of this theorem and hence the inverse matrix  $M^{-1}$  is  $q^{-1}$ -Manin. The matrices  $\tilde{A}$  and  $\tilde{D}$  are also  $q^{-1}$ -Manin as submatrices of  $M^{-1}$ . Lemma 4.8 implies that the matrices  $\tilde{A}$  and  $\tilde{D}$  are two-sided invertible. Then, writing down the formula (4.16) for  $I = J = (1, \dots, m)$  and  $I = J = (m+1, \dots, n)$  and taking into account (4.40) we obtain

$$\det_{q^{-1}} \tilde{A} = (\det_q M)^{-1} \det_q D, \quad \det_{q^{-1}} \tilde{D} = (\det_q M)^{-1} \det_q A. \quad (4.42)$$

This implies that the  $q^{-1}$ -determinants  $\det_{q^{-1}} \tilde{A}$  and  $\det_{q^{-1}} \tilde{D}$  are also two-sided invertible. Applying Theorem 4.7 to  $q^{-1}$ -Manin matrices  $\tilde{A}$  and  $\tilde{D}$  we conclude that their inverse  $\tilde{A}^{-1}$  and  $\tilde{D}^{-1}$ , i.e. the Schur complements (4.39), are  $q$ -Manin matrices. The formulae (4.41) follow from (4.42) if one takes into account the equality

$$(\det_{q^{-1}} \tilde{A})^{-1} = \det_q(\tilde{A}^{-1}) = \det_q(A - BD^{-1}C), \quad (4.43)$$

$$(\det_{q^{-1}} \tilde{D})^{-1} = \det_q(\tilde{D}^{-1}) = \det_q(D - CA^{-1}B), \quad (4.44)$$

(see the formula (4.34)). □

#### 4.2.4 Sylvester's theorem

Sylvester's identity is a classical determinantal identity (see, e.g. [20] theorem 1.5.3). The Sylvester identities for the non commutative case were discussed, in the framework of the theory of quasi-determinants, in [22], and, using combinatorial methods for the  $q$ -analogues of matrices of the form  $1 + M$  in [24]. Here we show that for  $q$ -Manin matrices the identity easily follows from the Schur Theorem 4.9. For the sake of simplicity, we shall suppose that all matrices are two-sided invertible as well as their determinants.

Let us first recall the commutative case: Let  $A$  be a matrix  $(a_{ij})_{m \times m}$ ; take  $n < i, j \leq m$ ; denote:

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{i*} = (a_{i1} \ a_{i2} \ \cdots \ a_{in}), \quad a_{*j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}. \quad (4.45)$$

Define the  $(m-n) \times (m-n)$  matrix  $B$  as follows:

$$B_{ij} = \det \begin{pmatrix} A_0 & a_{*j} \\ a_{i*} & a_{ij} \end{pmatrix}, \quad B = (B_{ij})_{n+1 \leq i, j \leq m}. \quad (4.46)$$

Then

$$\det B = \det A \cdot (\det A_0)^{m-n-1}. \quad (4.47)$$

**Theorem 4.10.** (*Sylvester's identity for  $q$ -Manin matrices.*) Let  $M$  be  $m \times m$  a  $q$ -Manin matrix with right and left inverse; take  $n < i, j \leq m$  and denote:

$$M_0 = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{pmatrix}, \quad M_{i*} = (M_{i1} \ M_{i2} \ \cdots \ M_{in}), \quad M_{*j} = \begin{pmatrix} M_{1j} \\ M_{2j} \\ \vdots \\ M_{nj} \end{pmatrix} \quad (4.48)$$

Define the  $(m-n) \times (m-n)$  matrix  $B$  as follows:

$$B_{ij} = (\det_q(M_0))^{-1} \cdot \det_q \begin{pmatrix} M_0 & M_{*j} \\ M_{i*} & M_{ij} \end{pmatrix}, \quad B = (B_{ij})_{n+1 \leq i, j \leq m}. \quad (4.49)$$

Then the matrix  $B$  is a  $q$ -Manin matrix and

$$\det_q B = (\det_q M_0)^{-1} \cdot \det_q M. \quad (4.50)$$

**Proof.** Once chosen  $M_0$ , we consider the resulting block decomposition of  $M$ ,

$$M = \begin{pmatrix} M_0 & M_1 \\ M_2 & M_3 \end{pmatrix}. \quad (4.51)$$

The key observation is that the matrix  $B$  defined by (4.49) equals to the Schur complement matrix:  $M_0 - M_2(M_3)^{-1}M_1$ . To see this, we use Schur complement Theorem (Theorem 4.9) again:

$$B_{ij} = (\det_q(M_0))^{-1} \cdot \det_q \begin{pmatrix} M_0 & M_{*j} \\ M_{i*} & M_{ij} \end{pmatrix} = (\det_q(M_0))^{-1} \cdot ((\det_q(M_0))(M_{ij} - M_{i*}M_0^{-1}M_{*j})) \quad (4.52)$$

$$(M_{ij} - M_{i*}M_0^{-1}M_{*j}) = (M_0 - M_2(M_3)^{-1}M_1) \quad (4.53)$$

In particular, we used the Schur formula  $\det_q(M) = \det_q(A)\det_q(D - CA^{-1}B)$  for blocks  $M_0 = A$ ,  $M_{*j} = B$ ,  $M_{i*} = C$ ,  $M_{ij} = D$ , the last being a  $1 \times 1$  matrix. The theorem now follows from the Schur property of the matrix  $B$  immediately; indeed,  $B = (M_0 - M_2(M_3)^{-1}M_1)$  is a  $q$ -Manin matrix, since a Schur complement is a  $q$ -Manin matrix by Theorem 4.9. Then  $\det_q B = (\det_q M_0)^{-1} \cdot \det_q M$  follows from the formula 4.41 for the determinant of Schur complements.  $\square$

### 4.3 Plücker relations: an example

In this subsection we shall briefly address the problem of the existence and of the form of Plücker relations for  $q$ -determinants of  $q$ -Manin matrices (giving an example). On general grounds (see [26]) we know that quasi-Plücker identities exists within the theory of quasi-determinants of Gel'fand-Retakh-Wilson. For  $q$ -Manin matrices, these identities acquire a form that (up to the appearance of suitable powers of  $q$ ), is the same as that of the commutative case. We shall only deal with the case of the Plücker identities in the case of  $q - Gr(2, 4)$ , which is sufficiently enlightening. The analysis of the generic case can be easily performed with some combinatorics.



**Proposition 4.11.** *Consider a  $4 \times 2$   $q$ -Manin matrix  $A$ , and let  $\pi_{ij}$  be the  $q$ -minors made from the  $i$ -th and  $j$ -th rows of  $A$ . Then it holds:*

$$(\pi_{12}\pi_{34} + q^{-4}\pi_{34}\pi_{12}) - (q^{-1}\pi_{13}\pi_{24} + q^{-3}\pi_{24}\pi_{13}) + q^{-2}(\pi_{14}\pi_{23} + \pi_{23}\pi_{14}) = 0 \quad (4.54)$$

**Proof.** The proof is basically the same as in the commutative case. Consider the  $q$ -Grassmann algebra  $\mathbb{C}[\psi_1, \dots, \psi_4]$ , and the variables  $\tilde{\psi}_1, \tilde{\psi}_2$  defined as:

$$(\tilde{\psi}_1, \tilde{\psi}_2) = (\psi_1, \psi_2, \psi_3, \psi_4) \cdot A. \quad (4.55)$$

It is clear that  $\tilde{\psi}_1 \wedge \tilde{\psi}_2 = \sum_{i \neq j} \pi_{ij} \psi_i \wedge \psi_j$ . By the defining property of  $q$ -Manin matrices in terms of coaction on  $q$ -Grassmann variables (Proposition 2.1),  $\tilde{\psi}_1$  and  $\tilde{\psi}_2$  are again Grassmann variables. Hence  $(\tilde{\psi}_1 \wedge \tilde{\psi}_2)^2$  must vanish. If we write explicitly this vanishing relation, taking into account the  $q$ -commutations between the  $\psi_j$ 's, we get the Plücker relations (4.54).  $\square$

**Remark.** Formula (4.54) can be compactly written as

$$\sum_{\sigma \in S'_4} q^{-\text{inv}(\sigma)} \pi_{\sigma(1)\sigma(2)} \pi_{\sigma(3)\sigma(4)} = 0,$$

that is the formula proven in ([26]) for the *quantum* matrix algebra, where  $S'_4$  is the subset of the group of permutations of four objects satisfying  $\sigma(1) < \sigma(2), \sigma(3) < \sigma(4)$ .

## 5 Tensor approach to $q$ -Manin matrices

In previous sections we mainly considered the  $q$ -Manin matrices over a non-commutative algebra  $\mathfrak{R}$  as homomorphisms from the corresponding Grassmann algebra to the tensor product of this algebra with  $\mathfrak{R}$ . Indeed, in Sections 3, 4 we have used formula  $\psi_j^M = \sum_{i=1}^n \psi_i M_{ij}$ . Another option is to work with them as with linear maps from the vector space  $\mathbb{C}^n$  to  $\mathfrak{R} \otimes \mathbb{C}^n$ , or in other words as with matrices over the non-commutative algebra  $\mathfrak{R}$ . In this case we shall interpret the  $q$ -determinant and  $q$ -minors in terms of certain higher tensors. Therefore one can call that *tensor approach*.

The approach used above is more natural to consider algebraic properties of  $q$ -Manin matrices, while the tensor approach is more useful in area of application to the quantum integrable systems. In this section we are going to reformulate some of the notions presented in previous Sections of the paper as well as to derive some new properties directly applicable to quantum integrable systems described by Lax matrices.

### 5.1 Leningrad tensor notations

Let us remind tensor notations known as Leningrad notations. First we shall identify  $ab$  with  $a \otimes b$  for  $a \in \mathfrak{R}$  and  $b \in \text{End}(\mathbb{C}^n)^{\otimes N}$ , where  $N$  is a number of tensor factors.<sup>6</sup> Let

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<sup>6</sup>Here we consider  $\mathfrak{R} \otimes \text{End}(\mathbb{C}^n)^{\otimes N}$  as a right  $\mathfrak{R}$ -module and a left  $\text{End}(\mathbb{C}^n)^{\otimes N}$ -module.

$\{X_\ell\}$  be a basis in the space  $\text{End}(\mathbb{C}^n)$  and  $C \in \mathfrak{R} \otimes \text{End}(\mathbb{C}^n)^{\otimes N}$  be a tensor over the algebra  $\mathfrak{R}$ . Then  $C$  can be written in the form

$$C = \sum_{\ell_1, \dots, \ell_N} C_{\ell_1, \dots, \ell_N} \cdot (X_{\ell_1} \otimes X_{\ell_2} \otimes \dots \otimes X_{\ell_N}), \quad (5.1)$$

where  $C_{\ell_1, \dots, \ell_N} \in \mathfrak{R}$  and  $X_{\ell_i} \in \text{End}(\mathbb{C}^n)$ ,  $i = 1, \dots, N$ . Introduce the notation  $C^{(k_1, \dots, k_N)}$  (Leningrad notation) for an element of  $\mathfrak{R} \otimes \text{End}(\mathbb{C}^n)^{\otimes N'}$ , where  $N' \geq N$ . It is defined as

$$C^{(k_1, \dots, k_N)} = \sum_{\ell_1, \dots, \ell_N} C_{\ell_1, \dots, \ell_N} \cdot (1 \otimes \dots \otimes 1 \otimes X_{\ell_1} \otimes 1 \otimes \dots \otimes 1 \otimes X_{\ell_2} \otimes 1 \otimes \dots \otimes 1 \otimes X_{\ell_N} \otimes 1 \otimes \dots \otimes 1), \quad (5.2)$$

where each  $X_{\ell_i}$  is placed in the  $k_i$ -th tensor factor (obviously  $1 \leq k_i \leq N'$ ). For example let  $N = N' = 2$ , and  $C = \sum_{\ell_1, \ell_2} C_{\ell_1, \ell_2} X_{\ell_1} \otimes X_{\ell_2}$ ; then

$$C^{(12)} = \sum_{\ell} C_{\ell_1, \ell_2} \cdot (X_{\ell_1} \otimes 1)(1 \otimes X_{\ell_2}), \quad C^{(21)} = \sum_{\ell} C_{\ell_1, \ell_2} \cdot (1 \otimes X_{\ell_1})(X_{\ell_2} \otimes 1). \quad (5.3)$$

Their product reads as

$$C^{(12)} C^{(21)} = \sum_{\ell_1, \ell_2, \ell'_1, \ell'_2} C_{\ell_1, \ell_2} \cdot C_{\ell'_1, \ell'_2} \cdot (X_{\ell_1} \otimes X_{\ell'_2}) \otimes (X_{\ell'_1} \otimes X_{\ell_2}). \quad (5.4)$$

Let  $E_{ij}$  be the standard matrices with entries  $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$ . Then the set  $\{E_{ij} \mid i, j = 1, \dots, n\}$  is a basis in  $\text{End}(\mathbb{C}^n)$  and each matrix  $M \in \mathfrak{R} \otimes \text{End}(\mathbb{C}^n)$  is decomposed as  $M = \sum_{i,j=1}^n M_{ij} E_{ij}$ , where  $M_{ij} \in \mathfrak{R}$  are entries of  $M$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis in  $\mathbb{C}^n$ :  $(e_i)^j = \delta_i^j$ , so that  $E_{ij} e_k = \delta_{kj} e_i$ . Then in this basis the action of the matrix  $M$  reads  $M e_j = \sum_{i=1}^n M_{ij} e_i$ . According to the Leningrad notation the tensor  $M^{(1)} M^{(2)} \dots M^{(m)}$  can be written as

$$M^{(1)} M^{(2)} \dots M^{(m)} = \sum_{\substack{i_1, \dots, i_m \\ j_1, \dots, j_m}} M_{i_1 j_1} M_{i_2 j_2} \dots M_{i_m j_m} \cdot (E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \dots \otimes E_{i_m j_m}). \quad (5.5)$$

## 5.2 Tensor relations for $q$ -Manin matrices

In this section we present the defining relations for Manin matrices in the tensor notations. We also present important higher order relations following from these defining quadratic relations.

### 5.2.1 Pyatov's Lemma

Let  $P^q \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$  be the  $q$ -permutation operator:  $P^q(e_i \otimes e_j) = q^{-\text{sgn}(i-j)} e_j \otimes e_i$ . The matrix of  $P^q$  (which we denote by the same symbol) can be written as

$$P^q = \sum_{i,j=1}^n q^{\text{sgn}(i-j)} E_{ij} \otimes E_{ji}. \quad (5.6)$$

As the usual permutation matrix  $P^1$  (when  $q = 1$ ) it satisfies  $(P^q)^2 = 1$ . Introduce the following tensors called  $q$ -anti-symmetrizer and  $q$ -symmetrizer:

$$A^q = \frac{1 - P^q}{2}, \quad S^q = \frac{1 + P^q}{2}. \quad (5.7)$$

Notice that these are orthogonal idempotents:  $(A^q)^2 = A^q$ ,  $(S^q)^2 = S^q$ ,  $A^q S^q = S^q A^q = 0$ .

The following lemma was suggested to us by P. Pyatov. Let  $M$  be a  $n \times n$  matrix with entries belonging to a associative algebra  $\mathfrak{R}$  over  $\mathbb{C}$ .

**Lemma 5.1.** (*Pyatov's lemma*). *Matrix  $M$  is a  $q$ -Manin matrix if and only if any of the following formulæ holds:*

$$M^{(1)} M^{(2)} - P^q M^{(1)} M^{(2)} P^q = P^q M^{(1)} M^{(2)} - M^{(1)} M^{(2)} P^q, \quad (5.8)$$

$$A^q M^{(1)} M^{(2)} A^q = A^q M^{(1)} M^{(2)}, \quad (5.9)$$

$$S^q M^{(1)} M^{(2)} S^q = M^{(1)} M^{(2)} S^q, \quad (5.10)$$

$$(1 - P^q) M^{(1)} M^{(2)} (1 + P^q) = 0. \quad (5.11)$$

**Proof.** It is easy to see that the matrix equations (5.8)–(5.11) are equivalent to each other. Rewriting, for instance, the equation (5.8) by entries one yields exactly the relations (2.3).  $\square$

Lemma 5.1 can be considered as a definition of  $q$ -Manin matrices in the tensor form. For instance one can construct  $q$ -Manin matrices from  $RLL$ -relations (see Section 6) using the relation (5.9) in the form given by the following

**Lemma 5.2.** *If the matrix  $M$  satisfy the equation*

$$A^q M^{(1)} M^{(2)} = T A^q \quad (5.12)$$

*for some matrix  $T \in \mathfrak{R} \otimes \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$  then  $M$  is a  $q$ -Manin matrix.*

**Proof.** Multiplying (5.12) by  $A^q$  from the right and taking into account the equality  $A^q A^q = A^q$  we obtain

$$A^q M^{(1)} M^{(2)} A^q = T A^q. \quad (5.13)$$

Using again the equation (5.12) we can substitute  $T A^q$  in the right hand side by  $A^q M^{(1)} M^{(2)}$ . So we derive (5.9) and hence  $M$  is a  $q$ -Manin matrix due to Lemma 5.1.  $\square$

**Lemma 5.3.** *Since  $(P^q)^{(-1)} = P^{q^{-1}}$  an  $n \times n$  matrix  $M$  is a  $q$ -Manin matrix if and only if any of these formulæ holds:*

$$M^{(2)} M^{(1)} - P^{q^{-1}} M^{(2)} M^{(1)} P^{q^{-1}} = P^{q^{-1}} M^{(2)} M^{(1)} - M^{(2)} M^{(1)} P^{q^{-1}}, \quad (5.14)$$

$$A^{q^{-1}} M^{(2)} M^{(1)} A^{q^{-1}} = A^{q^{-1}} M^{(2)} M^{(1)}, \quad (5.15)$$

$$S^{q^{-1}} M^{(2)} M^{(1)} S^{q^{-1}} = M^{(2)} M^{(1)} S^{q^{-1}}, \quad (5.16)$$

$$A^{q^{-1}} M^{(2)} M^{(1)} S^{q^{-1}} = 0. \quad (5.17)$$

Since  $(P^q)^\top = P^{q^{-1}}$  the transpose matrix  $M^\top$  is a  $q$ -Manin matrix if and only if any of the formulae holds:

$$M^{(1)}M^{(2)} - P^{q^{-1}}M^{(1)}M^{(2)}P^{q^{-1}} = M^{(1)}M^{(2)}P^{q^{-1}} - P^{q^{-1}}M^{(1)}M^{(2)}, \quad (5.18)$$

$$A^{q^{-1}}M^{(1)}M^{(2)}A^{q^{-1}} = M^{(1)}M^{(2)}A^{q^{-1}}, \quad (5.19)$$

$$S^{q^{-1}}M^{(1)}M^{(2)}S^{q^{-1}} = S^{q^{-1}}M^{(1)}M^{(2)}, \quad (5.20)$$

$$S^{q^{-1}}M^{(1)}M^{(2)}A^{q^{-1}} = 0, \quad (5.21)$$

$$M^{(2)}M^{(1)} - P^qM^{(2)}M^{(1)}P^q = M^{(2)}M^{(1)}P^q - P^qM^{(2)}M^{(1)}, \quad (5.22)$$

$$A^qM^{(2)}M^{(1)}A^q = M^{(2)}M^{(1)}A^q, \quad (5.23)$$

$$S^qM^{(2)}M^{(1)}S^q = S^qM^{(2)}M^{(1)}, \quad (5.24)$$

$$S^qM^{(2)}M^{(1)}A^q = 0. \quad (5.25)$$

### 5.2.2 Higher $q$ -(anti)-symmetrizers and $q$ -Manin matrices

Consider the group homomorphism  $\pi_q: \mathfrak{S}_m \rightarrow \text{Aut}((\mathbb{C}^n)^{\otimes m})$  defined on the generators of  $\mathfrak{S}_m$  by formula

$$\pi_q(\sigma_k) = (P^q)^{(k,k+1)} = \sum_{i,j=1}^n q^{\text{sgn}(i-j)} (1^{\otimes(k-1)} \otimes E_{ij} \otimes E_{ji} \otimes 1^{\otimes(m-k-1)}), \quad (5.26)$$

where  $\sigma_k = \sigma_{k,k+1}$  are adjacent permutations. This is a  $q$ -deformation of the standard representation of the symmetric group  $\mathfrak{S}_m$  on the space  $(\mathbb{C}^n)^{\otimes m}$ . The  $q$ -anti-symmetrizer and  $q$ -symmetrizer acting in the space  $(\mathbb{C}^n)^{\otimes m}$  reads

$$A_m^q = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} (-1)^\sigma \pi_q(\tau), \quad S_m^q = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} \pi_q(\tau), \quad (5.27)$$

where  $(-1)^\sigma = (-1)^{\text{inv}(\sigma)}$  is the sign of the permutation  $\sigma \in \mathfrak{S}_m$ . These formulae generalize the definitions (5.7) as  $A_2^q = A^q$ ,  $S_2^q = S^q$ .

We can also generalize the relations (5.9) and (5.10) for the higher  $q$ -anti-symmetrizers and  $q$ -symmetrizers.

**Theorem 5.4.** *Let  $M$  be an  $n \times n$   $q$ -Manin matrix. The tensor  $A_m^q M^{(1)} \dots M^{(m)}$  is invariant under multiplication by the  $q$ -anti-symmetrizer from the right:*

$$A_m^q M^{(1)} \dots M^{(m)} = A_m^q M^{(1)} \dots M^{(m)} A_m^q. \quad (5.28)$$

*The tensor  $M^{(1)} \dots M^{(m)} S_m^q$  is invariant under multiplication by the  $q$ -symmetrizer from the left:*

$$M^{(1)} \dots M^{(m)} S_m^q = S_m^q M^{(1)} \dots M^{(m)} S_m^q. \quad (5.29)$$

**Proof.** Let us note that the formula (5.28) is equivalent to the following proposition: the tensor  $A_m^q M^{(1)} \dots M^{(m)}$  is anti-invariant under multiplication by any element  $\tau \in \mathfrak{S}_m$  in the representation  $\pi_q$  from the right:

$$A_m^q M^{(1)} \dots M^{(m)} = (-1)^\tau A_m^q M^{(1)} \dots M^{(m)} \pi_q(\tau). \quad (5.30)$$

Indeed, since  $(-1)^\tau A_m^q \pi_q(\tau) = A_m^q$  the right hand side of (5.28) is invariant under multiplication by  $(-1)^\tau \pi_q(\tau)$  from the right and hence (5.30) holds. Conversely, summing the formula (5.30) over  $\tau \in \mathfrak{S}_m$  and dividing by  $m!$  yields (5.28).

Since the formula (5.28) have been already proved for  $m = 2$  (Lemma 5.1) we have the relation  $A^q M^{(1)} M^{(2)} (-P^q) = A^q M^{(1)} M^{(2)}$  or equivalently

$$(A^q)^{(k,k+1)} M^{(k)} M^{(k+1)} (-P^q)^{(k,k+1)} = (A^q)^{(k,k+1)} M^{(k)} M^{(k+1)}. \quad (5.31)$$

To prove the formula (5.30) for general  $m$  it is sufficient to prove it for the generators  $\tau = \sigma_k$ . In this case  $\pi_q(\tau) = (P^q)^{(k,k+1)}$ . Applying the formula  $A_m^q = A_m^q (A^q)^{(k,k+1)}$  in the right hand side of (5.30) and taking into account  $[(A^q)^{(k,k+1)}, M^{(l)}] = 0$ ,  $1 \leq l < k$ , and  $[M^{(l)}, (P^q)^{(k,k+1)}] = 0$ ,  $k+1 < l \leq m$ , we obtain the left hand side of (5.30).

The relation (5.29) is equivalent to the relations

$$M^{(1)} \dots M^{(m)} S_m^q = \pi_q(\tau) M^{(1)} \dots M^{(m)} S_m^q, \quad \tau \in \mathfrak{S}_m, \quad (5.32)$$

which can be proved in the same way.  $\square$

Analogously one can obtain the relations

$$A_m^{q^{-1}} M^{(m)} \dots M^{(1)} = A_m^{q^{-1}} M^{(m)} \dots M^{(1)} A_m^{q^{-1}}, \quad (5.33)$$

$$M^{(1)} \dots M^{(m)} S_m^{q^{-1}} = S_m^{q^{-1}} M^{(1)} \dots M^{(m)} S_m^{q^{-1}}, \quad (5.34)$$

where  $M$  is a  $q$ -Manin matrix. In particular, any  $q^{-1}$ -Manin matrix  $M$  satisfies

$$A_m^q M^{(m)} \dots M^{(1)} = A_m^q M^{(m)} \dots M^{(1)} A_m^q, \quad (5.35)$$

$$M^{(1)} \dots M^{(m)} S_m^q = S_m^q M^{(1)} \dots M^{(m)} S_m^q, \quad (5.36)$$

**Remark 5.1.** Let us remark that for  $m \geq 3$  the relations (5.28) and (5.29) are not equivalent and do not imply that  $M$  is a  $q$ -Manin matrix.

### 5.3 The $q$ -determinant and the $q$ -minors as tensor components

The vectors  $e_{j_1, \dots, j_m} = e_{j_1} \otimes \dots \otimes e_{j_m}$  form a basis of the space  $(\mathbb{C}^n)^{\otimes m}$ . Each tensor (over  $\mathbb{C}$  or  $\mathfrak{R}$ ) can be decomposed by entries with respect to this basis (and its dual). Here we obtain the relation between components of the tensor (5.28) and  $m \times m$   $q$ -minors the matrix  $M$ .

### 5.3.1 Action of the $q$ -anti-symmetrizer in the basis $\{e_{j_1, \dots, j_m}\}$

Let us consider the representation  $\sigma \mapsto (-1)^\sigma \pi_q(\sigma)$  of the permutation group  $\mathfrak{S}_m$  in terms of the basis  $\{e_{j_1, \dots, j_m}\}$ . For the adjacent transpositions  $\sigma_k$  its action has the form

$$-\pi_k(\sigma_k) e_{j_1, \dots, j_k, j_{k+1}, \dots, j_m} = -q^{\text{sgn}(j_{k+1}-j_k)} e_{j_1, \dots, j_{k+1}, j_k, \dots, j_m}, \quad (5.37)$$

(not  $(-q)^{\text{sgn}(j_{k+1}-j_k)}$ ). Then one can obtain the following formula

$$(-1)^{\tau^{-1}} \pi_q(\tau^{-1}) e_{k_{\sigma(1)}, \dots, k_{\sigma(m)}} = (-q)^{\text{inv}(\sigma\tau) - \text{inv}(\sigma)} e_{k_{\sigma\tau(1)}, \dots, k_{\sigma\tau(m)}}, \quad (5.38)$$

where  $k_1 < \dots < k_m$ ,  $\sigma, \tau \in \mathfrak{S}_m$ . It is proved in the same way as (3.7). The action of  $A_m^q$  on the basis elements  $e_{k_{\sigma(1)}, \dots, k_{\sigma(m)}}$  can be obtained by summing (5.38) over all  $\tau \in \mathfrak{S}_m$ , its action on the other elements of the basis is zero:

$$A_m^q e_{k_{\sigma(1)}, \dots, k_{\sigma(m)}} = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} (-q)^{\text{inv}(\sigma\tau) - \text{inv}(\sigma)} e_{k_{\sigma\tau(1)}, \dots, k_{\sigma\tau(m)}}, \quad A_m^q e_{\dots, i, \dots, i, \dots} = 0. \quad (5.39)$$

For the multi-indices  $I = (i_1, \dots, i_m)$  and  $J = (j_1, \dots, j_m)$  we will use the notation  $M_{j_1, \dots, j_m}^{i_1, \dots, i_m} = M_{IJ}$  for an  $n \times n$  matrix  $M$ , that is  $M_{j_1, \dots, j_m}^{i_1, \dots, i_m}$  is the  $m \times m$  matrix with entries  $(M_{j_1, \dots, j_m}^{i_1, \dots, i_m})_{kl} = M_{i_k, j_l}$ . This is the matrix consisting of the entries of the matrix  $M$  lying in the intersection of  $i_k$ -th row and  $j_l$ -th column. For example,  $M_j^i = M_{ij}$  are corresponding entries of  $M$ .

### 5.3.2 Components of the tensor $A_m^q M^{(1)} \dots M^{(m)}$

Let us denote by  $\{e^{i_1, \dots, i_m}\}$  the basis of  $((\mathbb{C})^{\otimes m})^*$  dual to the basis  $\{e_{j_1, \dots, j_m} = e_{j_1} \otimes \dots \otimes e_{j_m}\}$  of  $(\mathbb{C})^{\otimes m}$ , i.e.  $\langle e^{i_1, \dots, i_m}, e_{j_1, \dots, j_m} \rangle = \delta_{j_1}^{i_1} \dots \delta_{j_m}^{i_m}$ .

**Lemma 5.5.** *Let  $M$  be an arbitrary  $n \times n$  matrix (over an algebra  $\mathfrak{R}$ ), then the tensor  $A_m^q M^{(1)} \dots M^{(m)}$  has the following components*

$$\langle e^{\dots, i, \dots, i, \dots}, A_m^q M^{(1)} \dots M^{(m)} e_{j_1, \dots, j_m} \rangle = 0, \quad (5.40)$$

$$\langle e^{k_{\sigma(1)}, \dots, k_{\sigma(m)}}, A_m^q M^{(1)} \dots M^{(m)} e_{j_1, \dots, j_m} \rangle = \frac{1}{m!} (-q)^{\text{inv}(\sigma)} \det_q \left( M_{j_1, \dots, j_m}^{k_1, \dots, k_m} \right), \quad (5.41)$$

where  $k_1 < \dots < k_m$ ,  $\sigma \in \mathfrak{S}_m$ .

**Proof.** Substituting the action of the matrix  $M$  on the basis  $\{e_j\}$  in terms of its entries we obtain

$$\langle e^{i_1, \dots, i_m}, A_m^q M^{(1)} \dots M^{(m)} e_{j_1, \dots, j_m} \rangle = \sum_{s_1, \dots, s_m=1}^n M_{s_1, j_1} \dots M_{s_m, j_m} \langle e^{i_1, \dots, i_m}, A_m^q e_{s_1, \dots, s_m} \rangle. \quad (5.42)$$

If  $i_l = i_{l'} = i$  for some  $l \neq l'$  then the covector  $e^{i_1, \dots, i_m} = e^{\dots, i, \dots, i, \dots}$  is orthogonal to the image of the operator  $A_m^q$  (see the formula (5.39)), hence the corresponding components (5.42)

vanish – yielding the formula (5.40). Otherwise the indices  $i_1, \dots, i_m$  are pairwise different and can be represented as  $i_l = k_{\sigma(l)}$  for some  $\sigma \in \mathfrak{S}_m$  and  $k_1, \dots, k_m$  such that  $k_1 < \dots < k_m$ . On the other hand the action of  $A_m^q$  vanishes on the vectors  $e_{s_1, \dots, s_m}$  if  $s_l = s_{l'}$  for some  $l \neq l'$ . It means that the sum in (5.42) reduced in this case to the sum over the indices  $s_1, \dots, s_m$  that are represented as  $s_l = r_{\sigma'(l)}$  for some  $\sigma' \in \mathfrak{S}_m$  and  $r_1, \dots, r_m$  such that  $r_1 < \dots < r_m$ . Substituting  $i_l = k_{\sigma(l)}$  and  $s_l = r_{\sigma'(l)}$  to (5.42) and calculating the action of  $A_m^q$  on the basis vectors by the formula (5.38) we obtain

$$\begin{aligned}
& \langle e^{k_{\sigma(1)}, \dots, k_{\sigma(m)}}, A_m^q M^{(1)} \dots M^{(m)} e_{j_1, \dots, j_m} \rangle = \\
&= \frac{1}{m!} \sum_{\sigma', \tau \in \mathfrak{S}_m} \sum_{r_1 < \dots < r_m} M_{r_{\sigma'(1)}, j_1} \dots M_{r_{\sigma'(m)}, j_m} \langle e^{k_{\sigma(1)}, \dots, k_{\sigma(m)}}, (-1)^{\tau^{-1}} \pi_q(\tau^{-1}) e_{r_{\sigma'(1)}, \dots, r_{\sigma'(m)}} \rangle = \\
&= \frac{1}{m!} \sum_{\sigma', \tau \in \mathfrak{S}_m} \sum_{r_1 < \dots < r_m} M_{r_{\sigma'(1)}, j_1} \dots M_{r_{\sigma'(m)}, j_m} \times \\
&\quad \times \langle e^{k_{\sigma(1)}, \dots, k_{\sigma(m)}}, (-q)^{\text{inv}(\sigma'\tau) - \text{inv}(\sigma')} e_{r_{\sigma'\tau(1)}, \dots, r_{\sigma'\tau(m)}} \rangle = \\
&= \frac{1}{m!} \sum_{\sigma' \in \mathfrak{S}_m} (-q)^{\text{inv}(\sigma) - \text{inv}(\sigma')} M_{k_{\sigma'(1)}, j_1} \dots M_{k_{\sigma'(m)}, j_m} = \frac{1}{m!} (-q)^{\text{inv}(\sigma)} \det_q \left( M_{j_1, \dots, j_m}^{k_1, \dots, k_m} \right),
\end{aligned}$$

where we have used formula (3.1). □

**Corollary 5.5.1.** *For a  $q$ -Manin matrix  $M$  we obtain*

$$A_m^q M^{(1)} \dots M^{(m)} = m! \sum_{\substack{K=(k_1 < \dots < k_m) \\ J=(j_1 < \dots < j_m)}} \det_q(M_{KJ}) A_m^q (E_{k_1 j_1} \otimes \dots \otimes E_{k_m j_m}) A_m^q. \quad (5.43)$$

We denote by  $\text{tr}_{1, \dots, m}(\cdot)$  the trace over the space  $(\mathbb{C}^n)^{\otimes m}$ . Notice that this trace is the composition of the traces over each tensor factor:  $\text{tr}_{1, \dots, m}(\cdot) = \text{tr}_1 \text{tr}_2 \dots \text{tr}_m(\cdot)$ . The tensor  $A_m^q M^{(1)} \dots M^{(m)}$  can be considered as a matrix (over  $\mathfrak{R}$ ) acting on the space  $(\mathbb{C}^n)^{\otimes m}$ . So we can consider the trace of this matrix:

$$\text{tr}_{1, \dots, m} (A_m^q M^{(1)} \dots M^{(m)}) = \sum_{k_1, \dots, k_m=1}^n \langle e^{k_1, \dots, k_m}, A_m^q M^{(1)} \dots M^{(m)} e_{k_1, \dots, k_m} \rangle. \quad (5.44)$$

**Corollary 5.5.2.** *For an arbitrary  $n \times n$  matrix  $M$  the trace (5.44) has the form*

$$\text{tr}_{1, \dots, m} (A_m^q M^{(1)} \dots M^{(m)}) = \sum_{k_1 < \dots < k_m} \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} (-q)^{\text{inv}(\sigma)} \det_q \left( M_{k_1, \dots, k_m}^{k_1, \dots, k_m} \right)^\sigma, \quad (5.45)$$

where  $\left( M_{k_1, \dots, k_m}^{k_1, \dots, k_m} \right)^\sigma = M_{k_{\sigma(1)}, \dots, k_{\sigma(m)}}^{k_1, \dots, k_m}$  is the matrix  $M_{k_1, \dots, k_m}^{k_1, \dots, k_m}$  with columns permuted by  $\sigma$  (see the subsection 3.2).

In some cases we can rewrite the formula (5.45) using the Property 2 of Proposition 3.3.

**Corollary 5.5.3.** *For  $n \times n$  matrices  $M$  satisfying the relation (2.2) (in particular for  $q$ -Manin matrices) the trace of  $A_m^q M^{(1)} \cdots M^{(m)}$  is the sum of all principal  $m \times m$   $q$ -minors of this matrix:*

$$\mathrm{tr}_{1,\dots,m} (A_m^q M^{(1)} \cdots M^{(m)}) = \sum_{k_1 < \dots < k_m} \det_q (M_{k_1, \dots, k_m}^{k_1, \dots, k_m}); \quad (5.46)$$

in the notations of the Sections 3, 4 we can rewrite it as

$$\mathrm{tr}_{1,\dots,m} (A_m^q M^{(1)} \cdots M^{(m)}) = \sum_{K=(k_1 < \dots < k_m)} \det_q (M_{KK}), \quad (5.47)$$

In particular, the trace of  $A_n^q M^{(1)} \cdots M^{(n)}$  in this case is the  $q$ -determinant of this matrix:

$$\mathrm{tr}_{1,\dots,n} (A_n^q M^{(1)} \cdots M^{(n)}) = \det_q M. \quad (5.48)$$

**Corollary 5.5.4.** *Any  $q$ -Manin matrix  $M$  satisfies the relation*

$$A_n^q M^{(1)} \cdots M^{(n)} = A_n^q \det_q M. \quad (5.49)$$

**Proof.** Since the image of the idempotent operator  $A_n^q$  is one-dimensional the Proposition 5.4 for  $m = n$  implies  $A_n^q M^{(1)} \cdots M^{(n)} = \mathfrak{e}_n A_n^q$ , where  $\mathfrak{e}_n \in \mathfrak{R}$ . Using the formula  $\mathrm{tr}_{1,\dots,n}(A_n^q) = 1$  one gets  $\mathfrak{e}_n = \mathrm{tr}_{1,\dots,n} (A_n^q M^{(1)} \cdots M^{(n)}) = \det_q M$ .  $\square$

## 5.4 $q$ -powers of $q$ -Manin matrices. Cayley-Hamilton theorem and Newton identities

We remind the definition of "  $q$ -corrected" powers  $M^{[n]}$  for  $q$ -Manin matrices and show that the Cayley-Hamilton theorem and the Newton identities hold<sup>7</sup>

### 5.4.1 Cayley-Hamilton theorem

Remind that (see Definition 4) that the  $q$ -characteristic polynomial of an  $n \times n$   $q$ -Manin matrix  $M$  is the following linear combinations of the sums of principal  $q$ -minors:

$$\mathrm{char}_M(t) = \sum_{m=0}^n (-1)^m \mathfrak{e}_m t^{n-m}, \quad (5.50)$$

where  $\mathfrak{e}_0 = 1$ ,  $\mathfrak{e}_m = \sum_{i_1 < \dots < i_m} \det_q M_{i_1, \dots, i_m}^{i_1, \dots, i_m}$ .

**Definition 7.** *Let  $M$  be an  $n \times n$   $q$ -Manin matrix. We call  $q$ -powers of  $M$  the matrices  $M^{[m]}$  defined by the formulæ*

$$M^{[0]} = 1, \quad M^{[m]} = \mathrm{tr}_1 \left( P^q (M^{[m-1]})^{(1)} M^{(2)} \right). \quad (5.51)$$

---

<sup>7</sup>Compare for the rational case with [4, 5], and for the elliptic case with [39].



Using the notation  $A *_q B = \text{tr}_1 (P^q A^{(1)} B^{(2)}) = \sum_{ijk} q^{\text{sgn}(j-i)} A_{ij} B_{jk} E_{ik}$  one can write

$$M^{[m]} = M^{[m-1]} *_q M = (\dots ((M *_q M) *_q M) *_q \dots *_q M), \quad (5.52)$$

(it is a polynomial of order  $m$  in the matrix  $M$ ).

**Example 5.1.** The second  $q$ -power  $M^{[2]} = \text{tr}_1 (P^q M^{(1)} M^{(2)})$  looks as

$$M^{[2]} = \begin{pmatrix} a^2 + qbc & ab + db \\ ac + dc & d^2 + q^{-1}cb \end{pmatrix} = \begin{pmatrix} a^2 + qbc & ab + qbd \\ q^{-1}ca + dc & d^2 + q^{-1}cb \end{pmatrix}. \quad (5.53)$$

**Theorem 5.6.** (c.f. [15]) Any  $n \times n$   $q$ -Manin matrix annihilates its characteristic polynomial by its  $q$ -powers via right substitution:

$$\sum_{m=0}^n (-1)^m \mathbf{e}_m M^{[n-m]} = 0. \quad (5.54)$$

**Example 5.2.** The Cayley-Hamilton theorem in the case  $n = 2$  reads:

$$M^{[2]} - \text{tr}(M)M + \det_q(M)1_{2 \times 2} = 0. \quad (5.55)$$

**Proof.** Let us start with a proof of the following formulae

$$m \text{tr}_{1, \dots, m-1} (A_m^q M^{(1)} \dots M^{(m)}) = \quad (5.56)$$

$$= \mathbf{e}_{m-1} M - (m-1) \text{tr}_{1, \dots, m-2} (A_{m-1}^q M^{(1)} \dots M^{(m-1)}) *_q M. \quad (5.57)$$

Using  $A_m^q = \frac{1}{m} A_{m-1}^q (1 - (m-1)(P^q)^{(m-1, m)}) A_{m-1}^q$  and Corollary 5.5.3 yields

$$\begin{aligned} m \text{tr}_{1, \dots, m-1} (A_m^q M^{(1)} \dots M^{(m)}) &= \\ &= \mathbf{e}_{m-1} M - (m-1) \text{tr}_{1, \dots, m-1} (A_{m-1}^q (P^q)^{(m-1, m)} A_{m-1}^q M^{(1)} \dots M^{(m)}). \end{aligned} \quad (5.58)$$

In the (5.58) we can apply Proposition 5.4 for  $m-1$  and use periodicity of trace to present it in the form

$$\begin{aligned} - (m-1) \text{tr}_{1, \dots, m-1} ((P^q)^{(m-1, m)} A_{m-1}^q M^{(1)} \dots M^{(m)}) &= \\ = - (m-1) \text{tr}_m ((P^q)^{(m-1, m)} (\text{tr}_{1, \dots, m-2} A_{m-1}^q M^{(1)} \dots M^{(m-1)}) M^{(m)}) &= \\ = - (m-1) (\text{tr}_{1, \dots, m-2} A_{m-1}^q M^{(1)} \dots M^{(m-1)}) *_q M. \end{aligned} \quad (5.59)$$

So, the formula (5.56) is proven. Iterating it one obtains

$$m(-1)^{m-1} \text{tr}_{1, \dots, m-1} (A_m^q M^{(1)} \dots M^{(m)}) = \sum_{k=0}^{m-1} (-1)^k \mathbf{e}_k M^{[m-k]}. \quad (5.60)$$

The formula (5.54) is a particular case of (5.60) corresponding to  $m = n$ . Indeed, taking the trace over the first  $n-1$  spaces in the formula (5.49) and taking into account  $\text{tr}_{1, \dots, n-1} A_n^q = \frac{1}{n}$  we have  $\text{tr}_{1, \dots, n-1} A_n^q M^{(1)} \dots M^{(n)} = \frac{1}{n} \mathbf{e}_n$ .  $\square$

### 5.4.2 Newton theorem

The classical Newton formulae are relations between elementary symmetric functions  $\sum_{1 \leq i_1 < \dots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m}$  and sums of powers  $\sum_{k=1}^n \lambda_k^m$ . They allow to inductively express elementary symmetric functions via the sums of powers and vice versa. These functions can be presented as sums of principal minors and as traces of powers of the matrix  $\text{diag}(\lambda_1, \dots, \lambda_n)$ . The Newton formulae can be represented in the matrix form, which is valid for any square number matrix. In [4, 5] they were generalized for ( $q = 1$ ) Manin matrices. Here we present a version of the Newton formulae for  $q$ -Manin matrices.

**Theorem 5.7.** (c.f. [15]) *For any  $n \times n$   $q$ -Manin matrix  $M$  and any  $m \geq 0$  we have the relations*

$$m\epsilon_m = \sum_{k=0}^{m-1} (-1)^{m+k+1} \epsilon_k \text{tr}(M^{[m-k]}), \quad (5.61)$$

where  $\epsilon_m = 0$  for  $m > n$ .

**Proof.** Note that the formula (5.60) is still valid for  $m > n$  if we put  $\epsilon_k = 0$  for  $k > n$ . Taking the trace over the left space in this formula we obtain (5.61).  $\square$

**Example 5.3.** Let  $n = 2$ ,  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then  $\epsilon_0 = 1$ ,  $\epsilon_1 = \text{tr}(M^{[1]}) = \text{tr} M = a + d$ ,  $\epsilon_2 = \det_q M = ad - q^{-1}cb$ ,  $\epsilon_3 = \epsilon_4 = \dots = 0$ ,  $\text{tr}(M^{[2]}) = a^2 + qbc + d^2 + q^{-1}cb$ ,  $\text{tr}(M^{[3]}) = a^3 + qbca + qabc + qdbc + q^{-1}acb + q^{-1}dcb + d^3 + q^{-1}cbd$ . One can explicitly check the first tree (non-trivial) relations

$$\epsilon_1 = \text{tr}(M^{[1]}), \quad 2\epsilon_2 = -\text{tr}(M^{[2]}) + \epsilon_1 \text{tr}(M^{[1]}), \quad 0 = \text{tr}(M^{[3]}) - \epsilon_1 \text{tr}(M^{[2]}) + \epsilon_2 \text{tr}(M^{[1]}).$$

Using them one can express  $\text{tr}(M^{[3]})$  via  $\text{tr}(M^{[1]})$  and  $\text{tr}(M^{[2]})$  only:

$$\text{tr}(M^{[3]}) = \text{tr}(M^{[1]}) \text{tr}(M^{[2]}) + \frac{1}{2} \text{tr}(M^{[2]}) \text{tr}(M^{[1]}) - \frac{1}{2} (\text{tr}(M^{[1]}))^3. \quad (5.62)$$

Analogously one can express  $\text{tr}(M^{[4]})$ ,  $\text{tr}(M^{[5]})$  etc.

Let us now consider the first three relations for general  $n$

$$\epsilon_1 = \text{tr}(M^{[1]}), \quad \epsilon_2 = -\frac{1}{2} \text{tr}(M^{[2]}) + \frac{1}{2} \epsilon_1 \text{tr}(M^{[1]}), \quad (5.63)$$

$$\epsilon_3 = \frac{1}{3} \text{tr}(M^{[3]}) - \frac{1}{3} \epsilon_1 \text{tr}(M^{[2]}) + \frac{1}{3} \epsilon_2 \text{tr}(M^{[1]}). \quad (5.64)$$

Substituting them iteratively we can express  $\epsilon_1$ ,  $\epsilon_2$ ,  $\epsilon_3$  via traces of  $q$ -powers:

$$\epsilon_1 = \text{tr}(M^{[1]}), \quad (5.65)$$

$$\epsilon_2 = -\frac{1}{2} \text{tr}(M^{[2]}) + \frac{1}{2} (\text{tr}(M^{[1]}))^2, \quad (5.66)$$

$$\epsilon_3 = \frac{1}{3} \text{tr}(M^{[3]}) - \frac{1}{3} \text{tr}(M^{[1]}) \text{tr}(M^{[2]}) - \frac{1}{6} \text{tr}(M^{[2]}) \text{tr}(M^{[1]}) + \frac{1}{6} (\text{tr}(M^{[1]}))^3. \quad (5.67)$$

Thus iteratively substituting the first  $k$  relations (5.61) (divided by  $m$ ) we can express  $\mathfrak{e}_k$  via the traces of  $q$ -powers  $\text{tr}(M^{[1]}), \dots, \text{tr}(M^{[k]})$  for any  $k = 1, \dots, n$ . Note also that if  $m > n$  then the first  $m$  relations (5.61) allow to express  $\text{tr}(M^{[m]})$  via  $\text{tr}(M^{[1]}), \dots, \text{tr}(M^{[n]})$  as in the example.

Conversely, one can express the traces of  $q$ -powers of a  $q$ -Manin matrix via  $\mathfrak{e}_1, \dots, \mathfrak{e}_n$ . Indeed rewriting (5.61) as

$$\text{tr}(M^{[m]}) = (-1)^{m+1} m \mathfrak{e}_m + \sum_{k=1}^{m-1} (-1)^{k+1} \mathfrak{e}_k \text{tr}(M^{[m-k]}) \quad (5.68)$$

and after iterative substitutions one can express  $\text{tr}(M^{[m]})$  via  $\mathfrak{e}_1, \dots, \mathfrak{e}_m$ . For example the first three relations give us

$$\text{tr}(M^{[1]}) = \mathfrak{e}_1, \quad (5.69)$$

$$\text{tr}(M^{[2]}) = -2\mathfrak{e}_2 + \mathfrak{e}_1^2, \quad (5.70)$$

$$\text{tr}(M^{[3]}) = 3\mathfrak{e}_3 - 2\mathfrak{e}_1\mathfrak{e}_2 + \mathfrak{e}_1^3 - \mathfrak{e}_2\mathfrak{e}_1. \quad (5.71)$$

## 5.5 Inverse to a $q$ -Manin matrix

Here we obtain some tensor relations for the inverse to a  $q$ -Manin matrix and investigate the conditions when these relations holds. First consider a  $q$ -Manin matrix  $M$  over the algebra  $\mathfrak{R}$  invertible from the right with the  $q$ -determinant invertible from the left (as in Theorem 4.7). Then there exists a two-sided inverse  $M^{-1}$ , which is  $q^{-1}$ -Manin matrix. Due to Lemma 5.3 it satisfies

$$A^q(M^{-1})^{(2)}(M^{-1})^{(1)}A^q = A^q(M^{-1})^{(2)}(M^{-1})^{(1)}. \quad (5.72)$$

Suppose that  $M^{-1}$  is an  $n \times n$  matrix satisfying (5.72). According to Lemma 5.3 this is a  $q^{-1}$ -Manin matrix. In particular, it satisfies the relation (5.35), which takes the form

$$A_m^q(M^{-1})^{(m)} \dots (M^{-1})^{(1)} A_m^q = A_m^q(M^{-1})^{(m)} \dots (M^{-1})^{(1)}. \quad (5.73)$$

Also from Corollary 5.5.3 we obtain

$$\begin{aligned} \text{tr}_{1, \dots, m} \left( A_m^{q^{-1}}(M^{-1})^{(1)} \dots (M^{-1})^{(m)} \right) &= \text{tr}_{1, \dots, m} \left( A_m^q(M^{-1})^{(m)} \dots (M^{-1})^{(1)} \right) = \\ &= \sum_{K=(k_1 < \dots < k_m)} \det_{q^{-1}}(M_{KK}^{-1}), \end{aligned} \quad (5.74)$$

In particular one has

$$\text{tr}_{1, \dots, n} \left( A_n^q(M^{-1})^{(n)} \dots (M^{-1})^{(1)} \right) = \det_{q^{-1}}(M^{-1}). \quad (5.75)$$

Then taking the trace in the both hand sides of the relation

$$A_n^q(M^{-1})^{(n)} \dots (M^{-1})^{(1)} = \tilde{\mathfrak{e}}_n A_n^q, \quad (5.76)$$

where  $\tilde{\mathfrak{c}}_n \in \mathfrak{R}$ .

Using (5.73) we obtain

$$A_n^q (M^{-1})^{(n)} \cdots (M^{-1})^{(1)} = A_n^q \det_{q^{-1}}(M^{-1}). \quad (5.77)$$

**Proposition 5.8.** *Let  $M$  be  $n \times n$  right-invertible matrix over the algebra  $\mathfrak{R}$ . Assume that there exists a matrix  $B \in \mathfrak{R} \otimes \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$  such that*

$$BA^q M^{(1)} M^{(2)} = A^q. \quad (5.78)$$

*Then the right inverse  $M^{-1}$  is a  $q^{-1}$ -Manin matrix and satisfies consequently the relations (5.72)–(5.77).*

*If in addition  $M$  is a  $q$ -Manin matrix then its determinant is invertible from the right:*

$$\det_q M \det_{q^{-1}}(M^{-1}) = 1, . \quad (5.79)$$

*Moreover if  $M$  is invertible from the left in this case then its determinant is invertible from the left:*

$$\det_{q^{-1}}(M^{-1}) \det_q M = 1. \quad (5.80)$$

**Proof.** Multiplying (5.78) by  $(M^{-1})^{(2)}(M^{-1})^{(1)}$  from the right we obtain

$$BA^q = A^q (M^{-1})^{(2)} (M^{-1})^{(1)},$$

which implies (5.72) (c.f. Lemma 5.2). Hence  $M^{-1}$  is a  $q^{-1}$ -Manin matrix satisfying also (5.73)–(5.77). If  $M$  is  $q$ -Manin then the relations (5.28) and (5.49) hold. Multiplying (5.49) with (5.77) and using (5.28) we obtain  $A_n^q = A_n^q \det_q M \det_{q^{-1}}(M^{-1})$ , which implies (5.79). If  $M$  is invertible from the left then  $M^{-1}M = 1$  for any right inverse  $M^{-1}$ . Hence multiplying (5.77) with (5.49) and using (5.73) we obtain  $A_n^q = A_n^q \det_{q^{-1}}(M^{-1}) \det_q M$ , which implies (5.80).  $\square$

Conversely if the determinant of a  $q$ -Manin matrix  $M$  is left invertible then there exists a matrix  $B$  satisfying (5.78). It can be given in the form  $B = \sum_{\substack{i < j \\ k < l}} B_{kl}^{ij} (E_{ik} \otimes E_{jl} - q E_{jk} \otimes E_{il})$  where

$$B_{kl}^{ij} = \frac{1}{2} (-q)^{k+l-i-j} (\det_q M)^{-1} \det_{q^{-1}}(M_{KJ}), \quad (5.81)$$

$K = \setminus(kl)$ ,  $J = \setminus(ij)$ . One can check it using the Laplace expansion formula (3.34) for the case  $m = n - 2$  in the form

$$\sum_{k < l} (-q)^{k+l-i-j} \det_q M_{KJ} \det_q (M_{ab}^{ij}) = (\delta_a^i \delta_b^j - \delta_b^i \delta_a^j) \det_q M. \quad (5.82)$$

Thus for a two-sided invertible  $q$ -Manin matrix  $M$  the left invertibility of  $\det_q M$  is equivalent to the existence of the matrix  $B$  satisfying (5.78).

**Remark 5.2.** The condition (5.78) is satisfied for some  $B \in \mathfrak{R} \otimes \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$  if and only if there exists  $\tilde{B} \in \mathfrak{R} \otimes \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$  such that  $\tilde{B} A^q M^{(1)} M^{(2)} \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \setminus \{0\}$ . Indeed decomposing  $\tilde{B} A^q M^{(1)} M^{(2)}$  as  $\alpha A^q + \beta(1 - A^q)$  where  $\alpha, \beta \in \mathbb{C}$  and multiplying (5.9) by  $\tilde{B}$  from the left we obtain  $\beta = 0$ . Hence  $\alpha \neq 0$  and  $B = \tilde{B}/\alpha$  satisfies (5.78).

## 6 Integrable systems. $L$ -operators

### 6.1 $L$ -operators and $q$ -Manin matrices

Let us consider the trigonometric  $R$ -matrix  $R(z) \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)(z)$ <sup>8</sup> :

$$R(z) = \frac{1}{z-1} \left( (qz - q^{-1}) \sum_{i=1}^n E_{ii} \otimes E_{ii} + (z-1) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} (zE_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij}) \right), \quad (6.1)$$

where  $E_{ij}$  are basis elements of  $\text{End}(\mathbb{C}^n)$ :  $E_{ij}e_k = \delta_{jk}e_i$  and  $z$  is a complex parameter.

It is convenient sometimes to substitute  $z = u/v$ :

$$R(u/v) = \frac{qu - q^{-1}v}{u - v} \sum_{i=1}^n E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + \frac{q - q^{-1}}{u - v} \sum_{i < j} (uE_{ij} \otimes E_{ji} + vE_{ji} \otimes E_{ij}). \quad (6.2)$$

The  $R$ -matrix satisfies the Yang-Baxter equation

$$R^{(12)}(z_1/z_2)R^{(13)}(z_1/z_3)R^{(23)}(z_2/z_3) = R^{(23)}(z_2/z_3)R^{(13)}(z_1/z_3)R^{(12)}(z_1/z_2). \quad (6.3)$$

**Lemma 6.1.** *The  $R$ -matrix (6.1) at  $z = q^{-2}$  has the form*

$$R(q^{-2}) = 1 - P^q = 2A^q, \quad (6.4)$$

where  $P^q$  and  $A^q$  are defined by the formulae (5.6) and (5.7).

**Proof.** Substituting  $u = q^{-1}$ ,  $v = q$  to (6.2) one obtain

$$R(q^{-2}) = \sum_{i \neq j} (E_{ii} \otimes E_{jj} - q^{sgn(i-j)} E_{ji} \otimes E_{ij}) = 1 - P^q.$$

□

Consider an  $n \times n$  matrix  $L(z)$  with elements in a non-commutative algebra  $\mathfrak{R}$  depending on the parameter  $z$  (in general the entries of  $L(z)$  belong to  $\mathfrak{R}[[z, z^{-1}]]$ ) which satisfies  $RL$ -relation

$$R(z/w)L^{(1)}(z)L^{(2)}(w) = L^{(2)}(w)L^{(1)}(z)R(z/w), \quad (6.5)$$

where  $R(z/w)$  is the  $R$ -matrix (6.2). The matrix  $L(z)$  is called  $L$ -operator or *Lax matrix*. Notice that the Hopf algebra  $U_q(\widehat{\mathfrak{gl}}_n)$  can be described by a pair of  $L$ -operators such that the relations (6.5) for them are the defining commutation relation for this algebra.

The basic observation of the present Section is the following

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<sup>8</sup>Up to inessential changes the  $R$ -matrix is the same as in [40, 35] but slightly different from [1, 21].

**Proposition 6.2.** *Let  $L(z)$  be an  $L$ -operator satisfying (6.5). Then*

$$M = L(z)q^{2z\frac{\partial}{\partial z}} \quad (6.6)$$

*is a  $q$ -Manin matrix, where  $q^{2z\frac{\partial}{\partial z}}$  is the operator acting as  $q^{2z\frac{\partial}{\partial z}}f(z) = f(q^2z)$ . Also  $\widetilde{M} = L(z)^\top q^{-2z\frac{\partial}{\partial z}}$  is a  $q$ -Manin matrix, where  $L(z)^\top$  is the transpose of  $L(z)$ .*

**Proof.** Substituting  $w = q^2z$  to (6.5) and using (6.4) we obtain

$$A^q L^{(1)}(z) L^{(2)}(q^2z) = L^{(2)}(q^2z) L^{(1)}(z) A^q. \quad (6.7)$$

Multiplying (6.7) by the operator  $q^{4z\frac{\partial}{\partial z}}$  from the right and using  $f(z)\lambda^{z\partial_z} = \lambda^{z\partial_z}f(\lambda^{-1}z)$  one gets:

$$A^q L^{(1)}(z) q^{2z\frac{\partial}{\partial z}} L^{(2)}(z) q^{2z\frac{\partial}{\partial z}} = L^{(2)}(q^2z) L^{(1)}(z) q^{4z\frac{\partial}{\partial z}} A^q. \quad (6.8)$$

Then, applying Lemma 5.2 we conclude that the operator  $M = L(z)q^{2z\frac{\partial}{\partial z}}$  is  $q$ -Manin's matrix. Replacing  $z \rightarrow q^2z$  in the formula (6.7) one can analogously deduce that the matrix  $\widetilde{M}^\top = L(z)q^{-2z\frac{\partial}{\partial z}} = q^{-2z\frac{\partial}{\partial z}}L(q^2z)$  satisfies the relation (5.23).  $\square$

**Remark 6.1.** Similarly using (5.15) and (5.19) one can show that if an  $L$ -operator  $L(z)$  satisfies the relation

$$R(z/w)L^{(2)}(z)L^{(1)}(w) = L^{(1)}(w)L^{(2)}(z)R(z/w), \quad (6.9)$$

(which replaces (6.5)) then the matrices  $M = L(z)q^{-2z\frac{\partial}{\partial z}}$  and  $\widetilde{M} = L(z)^\top q^{2z\frac{\partial}{\partial z}}$  are  $q$ -Manin matrices.

## 6.2 Quantum determinant of $L$ -operator

Now we consider the quantum determinant of  $L$ -operators. They were introduced and extensively studied by L. Faddeev's school (Kulish, Sklyanin et al.) and play a fundamental role in the theory of affine quantum groups and corresponding quantum integrable systems. Remind that the quantum determinants of the  $L$ -operators of the Hopf algebra  $U_q(\widehat{\mathfrak{gl}}_n)$  describes the center of this algebra.

Consider an  $L$ -operator  $L(z)$  satisfying (6.5). The *quantum determinant* of  $L(z)$  is defined by the formula

$$\text{qdet } L(z) := \sum_{\tau \in \mathfrak{S}_n} (-q)^{-\text{inv}(\tau)} L_{\tau(1)1}(z) L_{\tau(2)2}(q^2z) \cdots L_{\tau(n)n}(q^{2(n-1)}z). \quad (6.10)$$

This notion is closely related with the notion of the  $q$ -determinant (3.1) of the associated  $q$ -Manin matrix  $L(z) q^{2nz\frac{\partial}{\partial z}}$ .

**Proposition 6.3.** *The  $q$ -determinant of  $M = L(z)q^{2z\frac{\partial}{\partial z}}$  satisfies*

$$\det_q M = \text{qdet } L(z) q^{2nz\frac{\partial}{\partial z}}. \quad (6.11)$$

**Proof.** The proposition is proven simply noticing that, for any permutation  $\tau \in \mathfrak{S}_n$ ,

$$\begin{aligned} & L_{\tau(1),1} q^{2z \frac{\partial}{\partial z}} \cdot L_{\tau(2),2} q^{2z \frac{\partial}{\partial z}} \cdots L_{\tau(n),n} q^{2z \frac{\partial}{\partial z}} = \\ & (-q)^{-\text{inv}(\tau)} L_{\tau(1)1}(z) L_{\tau(2)2}(q^2 z) \cdots L_{\tau(n)n}(q^{2(n-1)} z) q^{2nz \frac{\partial}{\partial z}}. \end{aligned} \quad (6.12)$$

□

**Remark.** The same result can be obtained using Corollary 5.5.4. Indeed, substituting  $M = L(z)q^{2z \frac{\partial}{\partial z}}$  in the formula (5.49) one gets

$$A_n^q L^{(1)}(z) L^{(2)}(q^2 z) \cdots L^{(n)}(q^{2(n-1)} z) = A_n^q \text{qdet } L(z). \quad (6.13)$$

More generally, consider multi-indices  $I = (i_1 < \dots < i_m)$  and  $J = (j_1 < \dots < j_m)$ . It follows from the structure of the  $R$ -matrix (6.1) that the submatrix  $L_{IJ}(z)$  of an  $L$ -operator  $L(z)$  also satisfies (6.5) with the  $m^2 \times m^2$   $R$ -matrix (6.1). The corresponding  $q$ -Manin matrix is the submatrix  $M_{IJ} = L_{IJ}(z)q^{2z \frac{\partial}{\partial z}}$  of the  $q$ -Manin matrix  $M = L(z)q^{2z \frac{\partial}{\partial z}}$ . So, the quantum minors of  $L(z)$  and the  $q$ -minors of  $M$  are related via

$$\begin{aligned} \det_q(M_{IJ}) &= \sum_{\tau \in \mathfrak{S}_m} (-q)^{-\text{inv}(\tau)} L_{i_{\tau(1)}j_1}(z) L_{i_{\tau(2)}j_2}(q^2 z) \cdots L_{i_{\tau(m)}j_m}(q^{2(m-1)} z) q^{2mz \frac{\partial}{\partial z}} = \\ &= \text{qdet } L_{IJ}(z) q^{2mz \frac{\partial}{\partial z}}. \end{aligned} \quad (6.14)$$

By virtue of Corollary 5.5.1 we can derive

$$\begin{aligned} A_m^q L^{(1)}(z) L^{(2)}(q^2 z) \cdots L^{(m)}(q^{2(m-1)} z) &= \\ &= m! \sum_{\substack{I=(i_1 < \dots < i_m) \\ J=(j_1 < \dots < j_m)}} \text{qdet } L_{IJ}(z) A_m^q (E_{i_1 j_1} \otimes \cdots \otimes E_{i_m j_m}) A_m^q. \end{aligned} \quad (6.15)$$

**Example 6.1.** As an application of this formulæ we consider the the Gauss decomposition of an  $L$ -operator:

$$L(z) = \begin{pmatrix} 1 & & F_{\alpha\beta}(z) \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} k_1(z) & & 0 \\ & \ddots & \\ 0 & & k_n(z) \end{pmatrix} \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ E_{\beta\alpha}(z) & & 1 \end{pmatrix} = \quad (6.16)$$

$$= \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ E'_{\alpha\beta}(z) & & 1 \end{pmatrix} \begin{pmatrix} k'_1(z) & & 0 \\ & \ddots & \\ 0 & & k'_n(z) \end{pmatrix} \begin{pmatrix} 1 & & F'_{\beta\alpha}(z) \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \quad (6.17)$$

If  $k_i(z)$  and  $k'_i(z)$  are invertible Propositions 3.7 and 6.3 imply

$$\text{qdet } L(z) = k_n(z) k_{n-1}(q^2 z) \cdots k_1(q^{2(n-1)} z) q^{-2nz \frac{\partial}{\partial z}} \quad (6.18)$$

$$= k'_1(z) k'_2(q^2 z) \cdots k'_n(q^{2(n-1)} z) q^{-2nz \frac{\partial}{\partial z}}. \quad (6.19)$$

### 6.3 Generating functions for the commuting operators

As it well known in the theory of quantum Yang-Baxter equations, the  $\mathfrak{R}$ -valued function

$$t_1(z) = \text{tr } L(z) \quad (6.20)$$

commute with itself for different values of the spectral parameters, i.e.,

$$t_1(z)t_1(w) = t_1(w)t_1(z), \quad \forall z, w \in \mathbb{C}. \quad (6.21)$$

It means that it generates the commuting operators:

$$t_1(z) = \sum_{k \in \mathbb{Z}} H_k z^k, \quad H_k H_l = H_l H_k. \quad (6.22)$$

Hence, whenever one of these operators is the Hamiltonian of a quantum system, we see that this Hamiltonian admits a family of commuting operators (in the classical limit, they are usually called constants of the motion). However, if the size  $n$  of the  $L$  operator is greater than two, then they are generally not enough to ensure simplicity of the joint spectrum (in the quantum case) and/or guarantee the applicability of the Arnol'd-Liouville theorem in the classical case. Here we construct the highest generating functions of commuting operators (integrals of motion) in terms of the q-Manin matrix  $M = L(z)q^{2z \frac{\partial}{\partial z}}$ .

Let us introduce the following matrix acting in the space  $(\mathbb{C}^n)^{\otimes m}$ :

$$\begin{aligned} \mathbb{R}_m(z_1, \dots, z_m) &= \overrightarrow{\prod}_{1 \leq i < j \leq m} R^{(i,j)}(z_i/z_j) = \\ &= (R^{(1,2)} R^{(1,3)} \dots R^{(1,m)}) \dots (R^{(m-2,m-1)} R^{(m-2,m)}) (R^{(m-1,m)}) = \\ &= (R^{(1,2)}) (R^{(1,3)} R^{(2,3)}) \dots (R^{(1,m)} \dots R^{(m-1,m)}), \end{aligned} \quad (6.23)$$

where  $R^{(i,j)} = R^{(i,j)}(z_i/z_j)$ . Note that one can rearrange it in the form

$$\mathbb{R}_m(z_1, \dots, z_m) = \mathbb{R}_k(z_1, \dots, z_k) \overrightarrow{\prod}_{1 \leq i \leq k} \overrightarrow{\prod}_{k+1 \leq j \leq m} R^{(i,j)}(z_i/z_j) \mathbb{R}_{m-k}^{(k+1, \dots, m)}(z_{k+1}, \dots, z_m), \quad (6.24)$$

where  $k \leq m$ . Using the Yang-Baxter equation (6.3) we can rewrite (6.24) in another order:

$$\begin{aligned} \mathbb{R}_m(z_1, \dots, z_m) &= \mathbb{R}_k(z_1, \dots, z_k) \mathbb{R}_{m-k}^{(k+1, \dots, m)}(z_{k+1}, \dots, z_m) \overrightarrow{\prod}_{1 \leq i \leq k} \overleftarrow{\prod}_{k+1 \leq j \leq m} R^{(i,j)}(z_i/z_j) = \\ &= \overleftarrow{\prod}_{1 \leq i \leq k} \overrightarrow{\prod}_{k+1 \leq j \leq m} R^{(i,j)}(z_i/z_j) \mathbb{R}_k(z_1, \dots, z_k) \mathbb{R}_{m-k}^{(k+1, \dots, m)}(z_{k+1}, \dots, z_m). \end{aligned} \quad (6.25)$$

There holds (see, e.g. [1]) the



**Lemma 6.4.**

$$\mathbb{R}_m(z, q^2 z, \dots, q^{2(m-k-1)} z) = m! A_m^q. \quad (6.26)$$

This lemma is proved in the Appendix A.

Substituting  $m = k + l$ ,  $z_i = q^{2(i-1)} z$  for  $i = 1, \dots, k$  and  $z_{k+i} = q^{2(i-1)} w$  for  $i = 1, \dots, l$  in (6.25) and taking into account (6.26) we obtain

$$A_{k,l}^q \mathbb{R}_{k,l}(z, w) = A_{k,l}^q \mathbb{R}_{k,l}(z, w) A_{k,l}^q, \quad (6.27)$$

where  $A_{k,l}^q = (A_k^q)^{(1, \dots, k)} (A_l^q)^{(k+1, \dots, k+l)}$  and

$$\mathbb{R}_{k,l}(z, w) = \overrightarrow{\prod}_{1 \leq i \leq k} \overleftarrow{\prod}_{k+1 \leq j \leq k+l} R^{(i,j)}(q^{2(i-j+k)} z/w). \quad (6.28)$$

Similarly one gets

$$A_{k,l}^q \mathbb{R}_{k,l}(z, w)^{-1} = A_{k,l}^q \mathbb{R}_{k,l}(z, w)^{-1} A_{k,l}^q. \quad (6.29)$$

Writing down the relation (5.28) for the  $q$ -Manin matrix  $M = L(z) q^{2z \frac{\partial}{\partial z}}$  we obtain

$$A_m^q L^{(1)}(z) L^{(2)}(q^2 z) \dots L^{(m)}(q^{2(m-1)} z) = A_m^q L^{(1)}(z) L^{(2)}(q^2 z) \dots L^{(m)}(q^{2(m-1)} z) A_m^q. \quad (6.30)$$

This implies

$$A_{k,l}^q \mathbb{L}_{k,l}(z, w) = A_{k,l}^q \mathbb{L}_{k,l}(z, w) A_{k,l}^q, \quad (6.31)$$

where

$$\begin{aligned} \mathbb{L}_{k,l}(z, w) &= \\ &= L^{(1)}(z) L^{(2)}(q^2 z) \dots L^{(k)}(q^{2(k-1)} z) L^{(k+1)}(w) L^{(k+2)}(q^2 w) \dots L^{(k+l)}(q^{2(l-1)} w). \end{aligned} \quad (6.32)$$

**Remark 6.2.** If  $L(z)$  is an  $L$ -operator satisfying (6.9) (not usual (6.5)) then the substitution of the  $q^{-1}$ -Manin matrix  $M = L(z) q^{-2z \frac{\partial}{\partial z}}$  to (5.35) yields

$$\begin{aligned} A_m^q L^{(m)}(z) L^{(m-1)}(q^{-2} z) \dots L^{(1)}(q^{-2(m-1)} z) &= \\ &= A_m^q L^{(m)}(z) L^{(m-1)}(q^{-2} z) \dots L^{(1)}(q^{-2(m-1)} z) A_m^q. \end{aligned} \quad (6.33)$$

**Proposition 6.5.** *If  $L(z)$  is an  $L$ -operator over  $\mathfrak{R}$  satisfying (6.5) then the  $\mathfrak{R}$ -valued functions*

$$t_k(z) = \text{tr}_{1, \dots, k} (A_k^q L^{(1)}(z) L^{(2)}(q^2 z) \dots L^{(k)}(q^{2(k-1)} z)) \quad (6.34)$$

*commute among themselves for different values of the spectral parameter:*

$$t_k(z) t_l(w) = t_l(w) t_k(z), \quad k, l = 1, 2, 3, \dots \quad (6.35)$$

**Proof.** Due to the  $RLL$ -relation (6.5) the products of  $L$ -operators (6.32) and

$$\widetilde{\mathbb{L}}_{k,l}(z, w) = L^{(k+l)}(q^{2(l-1)}w) \cdots L^{(k+2)}(q^2w) L^{(k+1)}(w) L^{(k)}(q^{2(k-1)}z) \cdots L^{(2)}(q^2z) L^{(1)}(z)$$

are related as

$$\mathbb{R}_{k,l}(z, w) \mathbb{L}_{k,l}(z, w) = \widetilde{\mathbb{L}}_{k,l}(z, w) \mathbb{R}_{k,l}(z, w). \quad (6.36)$$

Multiplying this equation by  $\mathbb{R}_{k,l}(z, w)^{-1}$  from the right, by  $A_{k,l}^q$  from the left and taking the trace over all the spaces we obtain the equality

$$\mathrm{tr}_{1,\dots,k+l} (A_{k,l}^q \mathbb{R}_{k,l}(z, w) \mathbb{L}_{k,l}(z, w) \mathbb{R}_{k,l}(z, w)^{-1}) = \mathrm{tr}_{1,\dots,k+l} (A_{k,l}^q \widetilde{\mathbb{L}}_{k,l}(z, w)). \quad (6.37)$$

Using the relations (6.27), (6.31), (6.29) and periodicity of the trace we rewrite the left hand side of (6.37) as

$$\begin{aligned} \mathrm{tr}_{1,\dots,k+l} (A_{k,l}^q \mathbb{R}_{k,l}(z, w) A_{k,l}^q \mathbb{L}_{k,l}(z, w) \mathbb{R}_{k,l}(z, w)^{-1}) &= \\ &= \mathrm{tr}_{1,\dots,k+l} (A_{k,l}^q \mathbb{R}_{k,l}(z, w) A_{k,l}^q \mathbb{L}_{k,l}(z, w) A_{k,l}^q \mathbb{R}_{k,l}(z, w)^{-1}) = \\ &= \mathrm{tr}_{1,\dots,k+l} (\mathbb{R}_{k,l}(z, w) A_{k,l}^q \mathbb{L}_{k,l}(z, w) A_{k,l}^q \mathbb{R}_{k,l}(z, w)^{-1} A_{k,l}^q) = \\ &= \mathrm{tr}_{1,\dots,k+l} (\mathbb{R}_{k,l}(z, w) A_{k,l}^q \mathbb{L}_{k,l}(z, w) A_{k,l}^q \mathbb{R}_{k,l}(z, w)^{-1}) = \\ &= \mathrm{tr}_{1,\dots,k+l} (A_{k,l}^q \mathbb{L}_{k,l}(z, w) A_{k,l}^q) = \mathrm{tr}_{1,\dots,k+l} (A_{k,l}^q \mathbb{L}_{k,l}(z, w)) = t_k(z) t_l(w). \end{aligned} \quad (6.38)$$

Since the right hand side of (6.37) equals  $t_l(w) t_k(z)$  we obtain (6.35).  $\square$

**Remark 6.3.** The relations (6.27), (6.29) follow from the relation (6.30) for the  $L$ -operators  $L^{(0)}(z) = \overleftarrow{\prod}_{1 \leq j \leq l} R^{(0,j)}(q^{2(1-j)}z/w)$ ,  $L^{(0)}(w) = \overleftarrow{\prod}_{1 \leq i \leq k} R^{(i,0)}(q^{2(i-1)}z/w)^{-1}$  (satisfying (6.5)) and from (6.33) for the  $L$ -operators  $L^{(0)}(w) = \overrightarrow{\prod}_{1 \leq i \leq k} R^{(i,0)}(q^{2(i-1)}z/w)$ ,  $L^{(0)}(z) = \overrightarrow{\prod}_{1 \leq j \leq l} R^{(0,j)}(q^{2(k-j)}z/w)^{-1}$  (satisfying (6.9)). Note that in this way we do not need Lemma 6.4 since we use the properties of  $q$ -Manin matrices instead.

The functions (6.34) are related to the sums of its principal  $q$ -minors of the  $q$ -Manin matrix  $M = L(z) q^{2z \frac{\partial}{\partial z}}$  by the formula

$$t_k(z) = \mathrm{tr}_{1,\dots,k} (A_k^q M^{(1)} \cdots M^{(k)}) q^{-2kz \frac{\partial}{\partial z}} = \sum_{I=(i_1 < \dots < i_m)} \det_q(M_{II}) q^{-2kz \frac{\partial}{\partial z}}. \quad (6.39)$$

The functions (6.34) can be regarded as a generated functions for the integrals of motion for the system defined by the Hamiltonians (6.22). They can be in turn gathered in the two-variable function

$$g(z, u) = \sum_{m=0}^n (-1)^m t_m(z) u^{n-m}. \quad (6.40)$$

It is related with the characteristic polynomial:

$$\mathrm{char}_M(u) = \sum_{m=0}^n (-1)^m t_m(z) q^{2mz \frac{\partial}{\partial z}} u^{n-m} = g(z, u q^{-2z \frac{\partial}{\partial z}}) q^{2nz \frac{\partial}{\partial z}}. \quad (6.41)$$

Comparing it with (5.50) we obtain

$$\mathfrak{e}_m = t_m(z) q^{2mz \frac{\partial}{\partial z}}. \quad (6.42)$$

Hence the functions (6.34) equal to sums of the corresponding quantum minors of  $L(z)$ :

$$t_m(z) = \sum_{I=(i_1 < \dots < i_m)} \text{qdet } L_{II}(z). \quad (6.43)$$

## 6.4 Quantum powers for $L$ -operators

Here we apply the Cayley-Hamilton theorem and Newton identities to the  $q$ -Manin matrix  $M = L(z) q^{2z \frac{\partial}{\partial z}}$ . In particular, the Newton identities give us a new family of generating functions of integrals of motions, which provide a suitable quantum version of the powers of classical  $L$ -operators.

Let us use the notation  $L^{[m]}(z)$  defined iteratively by the formulae

$$\begin{aligned} L^{[0]}(z) &= 1, & L^{[m]}(z) &= L^{[m-1]}(z) *_q L(q^{2(m-1)}z) = \\ &= (\dots ((L(z) *_q L(q^2z)) *_q L(q^4z)) *_q \dots *_q L(q^{2(m-1)}z)), \end{aligned} \quad (6.44)$$

(the notion of the  $*_q$ -product was introduced in Definition 7). This definition is related with the  $q$ -powers of  $M$ :

$$M^{[m]} = L^{[m]}(z) q^{2mz \frac{\partial}{\partial z}}. \quad (6.45)$$

Applying Theorems 5.6 and 5.7 for  $M = L(z) q^{2z \frac{\partial}{\partial z}}$  and taking into account the formulae (6.42), (6.45) one obtains

$$\sum_{m=0}^n (-1)^m t_m(z) L^{[n-m]}(q^{2m}z) = 0, \quad (6.46)$$

$$mt_m(z) = \sum_{k=0}^{m-1} (-1)^{m+k+1} t_k(z) \text{tr} (L^{[m-k]}(q^{2k}z)). \quad (6.47)$$

Let us consider the  $\mathfrak{R}$ -valued functions

$$I_k(z) = \text{tr} (L^{[k]}(z)), \quad k = 1, 2, \dots \quad (6.48)$$

Then the Newton identities (6.47) can be rewritten in terms of (6.48) as

$$mt_m(z) = \sum_{k=0}^{m-1} (-1)^{m+k+1} t_k(z) I_{m-k}(q^{2k}z). \quad (6.49)$$

As we can recurrently express the elements  $\mathfrak{e}_m$  and  $\text{tr}(M^{[k]})$  through each other using the Newton identities (5.61) the formula (6.49) allows us to express  $t_m(z)$  through  $I_k(z)$  and vice versa. This leads to the following.

**Theorem 6.6.** *The functions (6.48) commute with themselves, with each other and with the functions (6.34) for different values of parameters:*

$$I_k(z)I_l(w) = I_l(w)I_k(z), \quad (6.50)$$

$$I_k(z)t_l(w) = t_l(w)I_k(z), \quad (6.51)$$

where  $k, l \in \mathbb{Z}_{>0}$ .

**Proof.** Let us prove this theorem by induction. First we note that  $I_1(z) = t_1(z)$  and  $I_1(z)$  commute with itself (for different values of parameters) and with  $t_l(w)$ . Further suppose that  $I_1(z), I_2(z), \dots, I_{m-1}(z)$  commute with itself, each other and with  $t_l(w)$ . Since  $t_0(z) = 1$ , this allows to express the function  $I_m(z)$  through  $I_1(z), \dots, I_{m-1}(z)$  and  $t_l(w)$ ,  $l = 1, \dots, m$ :

$$I_m(z) = (-1)^{m+1} m t_m(z) - \sum_{k=1}^{m-1} (-1)^k t_k(z) I_{m-k}(q^{2k} z). \quad (6.52)$$

This implies that  $I_1(z), I_2(z), \dots, I_m(z)$  commute with the functions  $t_l(w)$  and with itself and each other for different  $z$ .  $\square$

The functions  $I_k(z)$  can be considered as generating another set of quantum integrals of motion for the system defined by  $t_l(z)$ . They form a set of integrals alternative to the set of integrals generated by the functions  $t_l(z)$ .

## A Proof of Lemma 6.4

For each  $s = 1, \dots, m-1$  one can move the factor  $R^{(s,s+1)}$  in the product (6.23) to the right using the Yang-Baxter equation (6.3). Since  $R(q^{-2}) = 2A_2^q$  this implies  $\mathbb{R}_m = -\mathbb{R}_m \pi_q(\sigma_s)$ , where  $\pi_q$  is the representation of symmetric group defined by (5.26). This means that for any  $\sigma \in \mathfrak{S}_m$  we have the identity  $\mathbb{R}_m = (-1)^\sigma \mathbb{R}_m \pi_q(\sigma)$ . In the other hand, the operator  $A_m^q$  satisfy the same identity  $A_m^q = (-1)^\sigma A_m^q \pi_q(\sigma)$ . Since  $\pi_q(\sigma_{sr}) e_{\dots, i, \dots, i, \dots} = 0$  (with subscripts  $i$  placed onto the  $s$ -th and  $r$ -th sites) we have  $\mathbb{R}_m e_{\dots, i, \dots, i, \dots} = 0 = m! A_m^q e_{\dots, i, \dots, i, \dots}$ . By the same reason taking into account the formula (5.38) we conclude that it is sufficient to check the equality (6.26) on the vectors  $e_{i_1, \dots, i_m}$  with  $i_1 < \dots < i_m$ . To do it we suppose by induction that the equality  $\mathbb{R}_{m-1} = (m-1)! A_{m-1}^q$  is already proven. Let us substitute the explicit expression for the  $R$ -matrices to the result of the action of  $\mathbb{R}_m$  written in the form

$$\begin{aligned} \mathbb{R}_m e_{i_1, \dots, i_m} &= \mathbb{R}_{m-1} R^{(1m)}(q^{2(1-m)}) R^{(2m)}(q^{2(2-m)}) \dots R^{(m-1,m)}(q^{-2}) e_{i_1, \dots, i_m} = \\ &= (m-1)! A_{m-1}^q R^{(1m)}(q^{2(1-m)}) R^{(2m)}(q^{2(2-m)}) \dots R^{(m-1,m)}(q^{-2}) e_{i_1, \dots, i_m}. \end{aligned} \quad (\text{A.1})$$

Due to the inequality  $i_1 < \dots < i_m$  each vector  $R^{(s+1,m)}(q^{2(s+1-m)}) \dots R^{(m-1,m)}(q^{-2}) e_{i_1, \dots, i_m}$  is a linear combinations of the vectors  $e_{j_1, \dots, j_m}$  where  $j_1, \dots, j_m$  are pairwise different and  $j_1 < \dots < j_s < j_m$ . Then we can proceed as follows. Substituting the expression (6.1) for  $R^{(sm)}(q^{2(s-m)})$  we see that the first sum and first term of third sum in (6.1) act

by zero, the second sum acts identically and the second term of the third sum acts as  $\frac{q - q^{-1}}{q^{2(s-m)} - 1} \pi_1(\sigma_{sm})$ , where  $\pi_1$  is a representation  $\pi_q$  at  $q = 1$ :

$$\pi_1(\sigma) e_{j_1, \dots, j_m} = e_{j_{\sigma^{-1}(1)}, \dots, j_{\sigma^{-1}(m)}}.$$

Thus, the expression (A.1) takes the form

$$(m-1)! A_{m-1}^q \left( 1 + \frac{q - q^{-1}}{q^{2(1-m)} - 1} \pi_1(\sigma_{1m}) \right) \cdots \left( 1 + \frac{q - q^{-1}}{q^{-2} - 1} \pi_1(\sigma_{m-1,m}) \right) e_{i_1, \dots, i_m}. \quad (\text{A.2})$$

Opening the big parentheses we obtain

$$(m-1)! A_{m-1}^q \left( 1 + \sum_{g=1}^{m-1} \sum_{1 \leq k_1 < \dots < k_g \leq m-1} \prod_{l=1}^g \frac{q - q^{-1}}{q^{2(k_l-m)} - 1} \pi_1(\sigma_{k_1 m} \cdots \sigma_{k_g m}) \right) e_{i_1, \dots, i_m}. \quad (\text{A.3})$$

Taking into account  $\pi_1(\sigma) e_{i_1, \dots, i_m} = (q)^{-\text{inv}(\sigma)} \pi_q(\sigma) e_{i_1, \dots, i_m}$  and  $\text{inv}(\sigma_{k_1 m} \cdots \sigma_{k_g m}) = 2m - 2k_1 - g$  and  $A_{m-1}^q \pi_q(\sigma_{k_1 m} \cdots \sigma_{k_g m}) = (-1)^{g-1} A_{m-1}^q \pi_q(\sigma_{k_1 m})$  we obtain

$$(m-1)! A_{m-1}^q \left( 1 - \sum_{g=1}^{m-1} \sum_{1 \leq k_1 < \dots < k_g \leq m-1} q^{2(k_1-m)} \prod_{l=1}^g \frac{1 - q^2}{q^{2(k_l-m)} - 1} \pi_q(\sigma_{k_1 m}) \right) e_{i_1, \dots, i_m}. \quad (\text{A.4})$$

The sum over  $g$  can be calculated as follows:

$$\begin{aligned} & \sum_{g=1}^{m-1} \sum_{1 \leq k_1 < \dots < k_g \leq m-1} q^{2(k_1-m)} \prod_{l=1}^g \frac{1 - q^2}{q^{2(k_l-m)} - 1} \pi_q(\sigma_{k_1 m}) = \\ &= \sum_{k_1=1}^{m-1} \pi_q(\sigma_{k_1 m}) q^{2(k_1-m)} \frac{1 - q^2}{q^{2(k_1-m)} - 1} \sum_{g=1}^{m-k_1} \sum_{k_1+1 \leq k_2 < \dots < k_g \leq m-1} \prod_{l=2}^g \frac{1 - q^2}{q^{2(k_l-m)} - 1} = \\ &= \sum_{k_1=1}^{m-1} \pi_q(\sigma_{k_1 m}) q^{2(k_1-m)} \frac{1 - q^2}{q^{2(k_1-m)} - 1} \prod_{s=k_1+1}^{m-1} \left( 1 + \frac{1 - q^2}{q^{2(s-m)} - 1} \right) = \sum_{k_1=1}^{m-1} \pi_q(\sigma_{k_1 m}). \quad (\text{A.5}) \end{aligned}$$

Finally, we have

$$\mathbb{R}_m e_{i_1, \dots, i_m} = (m-1)! A_{m-1}^q \left( 1 - \sum_{k=1}^{m-1} \pi_q(\sigma_{km}) \right) e_{i_1, \dots, i_m} = m! A_m^q e_{i_1, \dots, i_m}. \quad (\text{A.6})$$

Thus we have proved the formula (6.26) on all the basis  $\{e_{i_1, \dots, i_m}\}$ .

## B An alternative proof of the Lagrange-Desnanot-Jacobi-Lewis Carroll formula

Here we present an alternative proof of the Lagrange-Desnanot-Jacobi-Lewis Carroll formula expressed in the form (4.28)–(4.30) that demands the slightly weaker condition of right-invertibility on the  $q$ -Manin matrix  $M$ .

**Proposition B.1.** *Let  $M$  be a  $q$ -Manin matrix right-invertible. Then*

$$M_{i_1 j}^{adj} M_{i_2 j}^{-1} - q M_{i_2 j}^{adj} M_{i_1 j}^{-1} = 0, \quad (B.1)$$

$$M_{i_1 j_1}^{adj} M_{i_2 l_2}^{-1} - q M_{i_2 j_1}^{adj} M_{i_1 l_2}^{-1} = (-q)^{j_1 + l_2 - i_1 - i_2} \det_q (M_{\setminus(j_1 l_2) \setminus (i_1 i_2)}), \quad (B.2)$$

$$M_{i_1 l_2}^{adj} M_{i_2 j_1}^{-1} - q M_{i_2 l_2}^{adj} M_{i_1 j_1}^{-1} = (-q)^{j_1 + l_2 - i_1 - i_2 + 1} \det_q (M_{\setminus(j_1 l_2) \setminus (i_1 i_2)}), \quad (B.3)$$

where  $1 \leq i_1 < i_2 \leq n$ ,  $1 \leq j \leq n$  and  $1 \leq j_1 < j_2 \leq n$ .

**Proof.** Consider  $q$ -Grassmann variables  $\psi_i$ . Since the matrix  $M$  is  $q$ -Manin the variables  $\tilde{\psi}_j$  are also  $q$ -Grassmann. The right invertibility of  $M$  implies the equality  $\sum_{a=1}^n \tilde{\psi}_a M_{as}^{-1} = \psi_s$ . Multiplying it by  $\psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{i-1} \tilde{\psi}_{i+1} \cdots \tilde{\psi}_{k-1} \tilde{\psi}_{k+1} \cdots \tilde{\psi}_n$  from the left we obtain

$$\sum_{a=1}^n \psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{i-1} \tilde{\psi}_{i+1} \cdots \tilde{\psi}_{k-1} \tilde{\psi}_{k+1} \cdots \tilde{\psi}_n \tilde{\psi}_a M_{as}^{-1} = \psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{i-1} \tilde{\psi}_{i+1} \cdots \tilde{\psi}_{k-1} \tilde{\psi}_{k+1} \cdots \tilde{\psi}_n \psi_s. \quad (B.4)$$

The left hand side of this equation contains only two non-vanishing items, for  $a = k$  and  $a = i$ :

$$\begin{aligned} & \sum_{a=1}^n \psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{i-1} \tilde{\psi}_{i+1} \cdots \tilde{\psi}_{k-1} \tilde{\psi}_{k+1} \cdots \tilde{\psi}_n \tilde{\psi}_a M_{as}^{-1} = \\ & = (-q)^{k-n} \psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{i-1} \tilde{\psi}_{i+1} \cdots \tilde{\psi}_n M_{ks}^{-1} + (-q)^{i-n+1} \psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{k-1} \tilde{\psi}_{k+1} \cdots \tilde{\psi}_n M_{is}^{-1}. \end{aligned} \quad (B.5)$$

Due to the relation

$$\psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{i-1} \tilde{\psi}_{i+1} \cdots \tilde{\psi}_n = \psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_n \det_q M_{\setminus i}^r = \quad (B.6)$$

$$= (-q)^{i-r} \psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_n M_{ir}^{adj} \quad (B.7)$$

we can rewrite (B.5) as

$$(-q)^{i+k-r-n} \psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_n (M_{ir}^{adj} M_{ks}^{-1} - q M_{kr}^{adj} M_{is}^{-1}). \quad (B.8)$$

From the other hand, if  $r = s$  the right hand side of (B.4) vanish and, therefore, we obtain (B.1). Otherwise the right hand side of (B.4) is equal to

$$\begin{aligned} & \psi_r \tilde{\psi}_1 \cdots \tilde{\psi}_{i-1} \tilde{\psi}_{i+1} \cdots \tilde{\psi}_{k-1} \tilde{\psi}_{k+1} \cdots \tilde{\psi}_n \psi_s = \\ & = \psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_{s-1} \psi_{s+1} \cdots \psi_n \psi_s \det_q M_{\setminus ik}^{rs}. \end{aligned} \quad (B.9)$$

If  $r < s$  then

$$\psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_{s-1} \psi_{s+1} \cdots \psi_n \psi_s = (-q)^{s-n} \psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_n, \quad (B.10)$$

and substituting  $r = j$ ,  $s = l$  we obtain (B.2). If  $r > s$  then

$$\psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_{s-1} \psi_{s+1} \cdots \psi_n \psi_s = (-q)^{s-n+1} \psi_r \psi_1 \cdots \psi_{r-1} \psi_{r+1} \cdots \psi_n, \quad (B.11)$$

and substituting  $r = l$ ,  $s = j$  we obtain (B.3).  $\square$

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